The only other alternative is that \( \Delta \neq 0 \), which is the conclusion we desired.

The method of induction is concerned with proving an infinite number of statements \( s_1, s_2, \ldots \), one for each positive integer \( n \). If \( s_1 \) is true, and if for any positive integer \( k \) the statement \( s_k \) implies the statement \( s_{k+1} \), then all the statements \( s_1, s_2, \ldots \), are true. An example of a result which can be proved using induction is the formula

\[
(fg)^{(k)} = \sum_{i=0}^{k} \binom{k}{i} f^{(k-i)} g^{(i)}, \quad \binom{k}{i} = \frac{k!}{i!(k-i)!},
\]

for the \( k \)-th derivative of the product of two complex-valued functions \( f, g \) which have \( k \) derivatives; see (4.3). The proof is the same as the induction used to prove the binomial formula

\[
(a + b)^k = \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i, \quad (k = 1, 2, \ldots),
\]

for the powers of the sum of two complex numbers \( a, b \). The method of induction is equivalent to a property of the positive integers, and consequently we assume that this method is a valid method of proof.

The principles of discovery (a), (b), and the methods of proof (i), (ii), (iii), will be used many times throughout this book. It will be instructive for the student to identify which principles and methods are being used in any particular situation.

CHAPTER 1

Introduction—Linear Equations of the First Order

1. Introduction

In Sec. 2 we discuss what is meant by an ordinary differential equation and its solutions. Various problems which arise in connection with differential equations are considered in Sec. 3, notably initial value problems, boundary value problems, and the qualitative behavior of solutions. In a succession of easy steps we solve the linear equation of the first order in Secs. 4–7.

2. Differential equations

Suppose \( f \) is a complex-valued function defined for all real \( z \) in an interval \( I \), and for complex \( y \) in some set \( S \). The value of \( f \) at \((x, y)\) is denoted by \( f(x, y) \). An important problem associated with \( f \) is to find a (complex-valued) function \( \phi \) on \( I \), which is differentiable there, such that for all \( x \) on \( I \),

\[
\begin{align*}
(i) \quad & \phi(x) \text{ is in } S, \\
(ii) \quad & \psi(x) = f(x, \phi(x)).
\end{align*}
\]

This problem is called an ordinary differential equation of the first order, and is denoted by

\[
y' = f(x, y).
\]

(2.1)

The ordinary refers to the fact that only ordinary derivatives enter into the problem, and not partial derivatives. If such a function \( \phi \) exists on \( I \) satisfying (i) and (ii) there, then \( \phi \) is called a solution of (2.1) on \( I \).
As an example consider the case when \( f \) is independent of \( y \), that is, we have the equation
\[
y' = f(x),
\]
(2.2)
where \( f \) is defined on some interval \( I \). The problem is to find a function \( \phi \) on \( I \) such that \( \phi' \) exists there, and \( \phi'(x) = f(x) \). This is one of the most important problems considered in the study of calculus. Indeed, if \( f \) is \textit{continuous} on \( I \), we know that the indefinite integral function \( \phi_0 \) defined by
\[
\phi_0(x) = \int_{x_0}^{x} f(t) \, dt,
\]
where \( x_0 \) is some fixed point in \( I \), is a solution of (2.2). Moreover, if \( \phi \) is any solution of (2.2), then there is a constant \( c \) such that
\[
\phi(x) = \phi_0(x) + c
\]
for all \( x \) in \( I \); and every constant \( c \) gives rise to a solution in this way. Thus all solutions of (2.2) are known in case \( f \) is continuous on \( I \), and the study of (2.2) reduces to the study of integration.

For a second example, suppose that \( \phi(x) \) denotes the amount of a certain substance at time \( x \), and we know that the substance increases at a rate proportional to the amount present at any time \( x \). Then we must have
\[
\phi'(x) = k\phi(x),
\]
where \( k \) is some constant. Thus \( \phi \) is a solution of the differential equation
\[
y' = ky.
\]
(2.3)
Conventional examples of processes described by this equation are population growth (\( k > 0 \)) and radioactive decay (\( k < 0 \)). A solution of (2.3) is given by
\[
\phi(x) = e^{kx},
\]
which exists for all real \( x \).

The problem posed by the equation \( y' = f(x, y) \) has a simple geometrical interpretation in case \( f \) is real-valued, and \( y \) is defined on a set \( S \) of real numbers. Then for each \( x \) in \( I \) and \( y \) in \( S \) we are given a number \( f(x, y) \), which may be thought of as the slope of a straight line through the point \( (x, y) \). A solution of \( y' = f(x, y) \) on \( I \) is a function \( \phi \) whose graph (the set of points \( (x, \phi(x)) \), \( x \) in \( I \)) is a curve whose tangent at \( (x, \phi(x)) \) has the slope \( \phi'(x) \), which is the same as the given slope \( f(x, \phi(x)) \) at this point. Thus, geometrically we are given a set of directions, and the differential equation is the problem of finding curves having these directions as tangents. The set of directions \( \{f(x, y)\} \) is called a \textit{direction field}. Fig. 3 shows such a field for \( f(x, y) = -xy \), and the curve sketched is the solution \( \phi(x) = 2e^{-(x^2/2)} \) of the equation \( y' = -xy \).

Figure 3. The direction field given by \( f(x, y) = -xy \) (\( y > 0 \))

Sometimes a differential equation occurs in a slightly more general form, where the derivative \( y' \) is not by itself on one side of the equation. Thus it might be necessary to consider an equation of the form
\[
F(x, y, y') = 0.
\]
(2.4)
Here \( F \) is some function defined for real \( x \) in an interval \( I \), and complex \( y, y_0 \) in sets \( S_1, S_2 \) respectively. Then (2.4) is the problem of finding a (complex-valued) function \( \phi \) on \( I \), which is differentiable there, such that for all \( x \) on \( I \),
(i) \( \phi(x) \) is in \( S_1 \), \( \phi'(x) \) is in \( S_2 \)

(so that \( F(x, \phi(x), \phi'(x)) \) is defined),
(ii) \( F(x, \phi(x), \phi'(x)) = 0 \).

This problem is also called an \textit{ordinary differential equation of the first order}. The equation (2.1) is the special case when
\[
F(x, y, y') = y' - f(x, y).
\]
Usually we shall consider equations in the form (2.1), since it can be shown that (2.4) can be reduced to the form (2.1) under rather general conditions on \( F \).

More general differential equations involve higher order derivatives. Let \( F \) now be a function defined for real \( x \) in an interval \( I \), and for complex \( y_1, y_2, \ldots, y_{n+1} \) in sets \( S_1, S_2, \ldots, S_{n+1} \) respectively. The problem of finding a function \( \phi \) on \( I \), having \( n \) derivatives there, and such that for all \( x \) in \( I \),
(i) \( \phi^{(k-n)}(x) \) is in \( S_k \) \quad (k = 1, 2, \ldots, n + 1),
(\( \phi^{(0)}(x) = \phi(x) \)),
(ii) \( F(x, \phi(x), \phi'(x), \ldots, \phi^{(n)}(x)) = 0 \),
is called an ordinary differential equation of the \( n \)th order, and is denoted by

\[
F(x, y, y', \cdots, y^{(n)}) = 0.
\]  

(2.5)

A function \( \phi \) on \( I \), with \( n \) derivatives, satisfying (i) and (ii) is called a solution of (2.5) on \( I \). Again it will be the usual situation to consider those equations of the form

\[
y^{(n)} = f(x, y, y', \cdots, y^{(n-1)}).
\]

An example of a second order equation is

\[
y'' + y = 0,
\]  

(2.6)

which arises naturally in the study of electrical and mechanical oscillations. Two solutions \( \phi_1, \phi_2 \) which exist for all real \( x \) are given by

\[
\phi_1(x) = \cos x, \quad \phi_2(x) = \sin x.
\]

3. Problems associated with differential equations

When presented with a differential equation our first impulse might be to try to find all solutions of it. Ideally we would like to write down these solutions in terms of well-known functions. This can be done for a large number of very important equations. For example, we indicated in Sec. 2 that all solutions of

\[
y' = f(x), \quad (f \text{ continuous}),
\]  

(3.1)

are given by

\[
\phi(x) = \int_{x_0}^x f(t) \, dt + c,
\]  

(3.2)

where \( x_0 \) is some point in the interval where \( f \) is defined, and \( c \) is any constant. All solutions of

\[
y' = ky
\]  

(3.3)

are of the form

\[
\phi(x) = ce^{kt},
\]  

(3.4)

and \( c \) can be any constant. We shall prove this elementary fact in Sec. 5. Also every solution of

\[
y'' + y = 0
\]  

(3.5)

has the form

\[
\phi(x) = c_1 \cos x + c_2 \sin x,
\]  

(3.6)

where \( c_1, c_2 \) may be arbitrary constants. The proof of this will occur in Chap. 2.

Sec. 3

Introduction—Linear Equations of the First Order

Frequently we are not interested in all solutions of an equation, but only those satisfying certain other conditions. These conditions may take many forms, but two of the most important types are initial conditions and boundary conditions. An initial condition is a condition on the solution at one point. For example, the solution of (3.3) having the property that \( \phi(0) = 2 \) (the initial condition) is readily seen from (3.4) to be given by

\[
\phi(x) = 2e^{2x}.
\]

Such an initial value problem would be denoted by

\[
y' = ky, \quad y(0) = 2.
\]

Similarly, the solution \( \phi \) of (3.5) satisfying

\[
\phi(0) = 1, \quad \phi'(0) = 2,
\]

is given by

\[
\phi(x) = \cos x + 2 \sin x.
\]

This problem would be denoted by

\[
y'' + y = 0, \quad y(0) = 1, \quad y'(0) = 2.
\]

A boundary condition is a condition on the solution at two or more points. For example, the solution \( \phi \) of (3.5) satisfying

\[
\phi(0) = 1, \quad \phi'(2\pi) = -1,
\]

is given by

\[
\phi(x) = \cos x - \sin x.
\]

There are many equations for which it is not obvious that solutions exist at all; and if they do, it might not be possible to write down "nice" formulas for them. For example, consider the equation

\[
y'' + y' + \sin y = 0,
\]  

(3.7)

which is encountered in the study of the motion of a pendulum. It can be shown that (3.7) has solutions, satisfying any given real initial condition, which exist for all real \( x \), although we can not express them in terms of functions we meet in calculus. How do we solve equations such as (3.7), that is, find the solutions? One method is to develop mathematical procedures which would allow us to compute the value of a solution at any given \( x \) to any desired degree of accuracy. This method should be sufficiently general to cover a large number of equations. Such a procedure is developed in Chap. 5, where a general method for computing solutions to initial value problems is given.

Even though it is impossible to express solutions of some equations in nice formulas, it is often the case that we can say a good deal about the properties of the solutions. In many situations it is just some property of
the solutions which we wish to investigate. For example, without solving (3.7) we can show that any solution \( \phi \) for which
\[
-\pi < \phi(0) < \pi, \quad \phi'(0) = 0,
\]
will tend to zero as \( x \to \infty \). This corresponds to the fact that the oscillations of the pendulum are damped, and eventually the pendulum will stay arbitrarily close to its equilibrium position \( y = 0 \).

**EXERCISES**

1. Find all solutions of the following equations on \(-\infty < x < \infty\):
   (a) \( y' = e^x + \sin x \)
   (b) \( y'' = 2 + x \)
   (c) \( y^{(k)} = 0, \) (k a positive integer)
   (d) \( y'' = x^2 \)

2. Verify that the following are solutions of the differential equations given:
   (a) \( \phi(x) = e^{-x^2} \) for \( y'' + (\cos x)y = 0 \)
   (b) \( \phi(x) = \sin x - 1, \) for \( y' + (\cos x)y = \sin x \cos x \)
   (c) \( \phi(x) = 1, \) for \( y'' - y' = 0 \)
   (d) \( \phi(x) = e^x, \) for \( y'' - y' = 0 \)
   (e) \( \phi(x) = c_1 + c_2 e^x, \) for \( y'' - y' = 0, \) \( (c_1, c_2 \) any constants)
   (f) \( \phi(x) = \sin 2x, \) for \( y'' + 4y = 0 \)
   (g) \( \phi(x) = e^{2x}, \) for \( y'' + 4y = 0 \)
   (h) \( \phi(x) = c_1 \cos kx + c_2 \sin kx, \) for \( y'' + k^2y = 0, \) \( (k a \) positive constant, and \( c_1, c_2 \) any constants)

3. Consider the equation \( y'' + 5y = 2 \).
   (a) Show that the function \( \phi \) given by
   \[
   \phi(x) = \frac{1}{2} + ce^{-5x}
   \]
   is a solution, where \( c \) is any constant.
   (b) Assuming every solution has this form, find that solution satisfying \( \phi(1) = 2 \).
   (c) Find that solution satisfying \( \phi(1) = 3\phi(0) \).

4. Consider the equation \( y'' = 3x + 1 \).
   (a) Find all solutions on the interval \( 0 \leq x \leq 1 \).
   (b) Find that solution \( \phi \) which satisfies \( \phi(0) = 1, \phi'(0) = 2 \).
   (c) Find that solution \( \phi \) which satisfies \( \phi(0) = 0, \phi(1) = 3 \).

5. Consider the equation \( y' = ky \) on \(-\infty < x < \infty \), where \( k \) is some constant.
   (a) Show that if \( \phi \) is any solution, and \( \psi(x) = \phi(x)e^{-kx} \), then \( \psi(x) = c \), where \( c \) is a constant. (Hint: Show that \( \psi(x) = 0 \) for all \( x \).)
   (b) Prove that if \( \text{Re} \ k < 0 \) then every solution tends to zero as \( x \to \infty \).
   (c) Prove that if \( \text{Re} \ k > 0 \) then the magnitude of every non-trivial (not identically zero) solution tends to \( \infty \) as \( x \to \infty \).
   (d) What can you say about the magnitudes of the solutions if \( \text{Re} \ k = 0 \)?

**Sec. 4**

**Introduction—Linear Equations of the First Order**

4. **Linear equations of the first order**

We initiate our study of differential equations by considering the simple case of a **linear equation of the first order**. This is an equation of the form
\[
y' + a(x)y = b(x),
\]
where \( a, b \) are certain functions defined on an interval \( I \). Writing this in the form \( y' = f(x, y) \) we see that
\[
f(x, y) = -a(x)y + b(x).
\]

If \( b(x) = 0 \) for all \( x \) in \( I \), the corresponding equation
\[
y' + a(x)y = 0
\]
is called a **homogeneous equation**, whereas if \( b \) is not identically zero on \( I \), (4.1) is called a non-homogeneous equation.

We note that if \( b(x) = 0 \) for all \( x \) in \( I \), then the \( f \) of (4.2) is linear in \( y \), that is,
\[
f(x, y_1 + y_2) = f(x, y_1) + f(x, y_2),
\]
and homogeneous in \( y \), that is,
\[
f(x, cy) = cf(x, y),
\]
where \( c \) is any constant.

We first solve the simple case of (4.1) when \( a \) is a constant, and then treat the more general case.

5. **The equation** \( y' + ay = 0 \)

If \( a \) is a constant and \( \phi \) is a solution of
\[
y' + ay = 0,
\]
then \( \phi' + a\phi = 0 \), and this implies that
\[
e^{ax}(\phi' + a\phi) = 0,
\]
or
\[
(e^{ax}\phi)' = 0.
\]
Therefore there is a constant \( c \) such that \( e^{ax}\phi(x) = c \), or
\[
\phi(x) = ce^{-ax}.
\]
We have shown that any solution \( \phi \) of (5.1) must have the form (5.2), where \( c \) is some constant. Conversely, if \( c \) is any constant, the function \( \phi \) defined by (5.2) is a solution of (5.1), for
\[
\phi'(x) + a\phi(x) = -ace^{-as} + ace^{-as} = 0.
\]
We have proved a small theorem.

**Theorem 1.** Consider the equation
\[
y' + ay = 0,
\]
where \( a \) is a complex constant. If \( c \) is any complex number, the function \( \phi \) defined by
\[
\phi(x) = ce^{-as}
\]
is a solution of this equation, and moreover every solution has this form.

Notice that all solutions exist for all real \( x \), that is, for \(-\infty < x < \infty\). Also note that the constant \( c \) is the value of \( \phi \) at 0, that is, \( c = \phi(0) \).

6. The equation \( y' + ay = b(x) \)

Let \( a \) be a constant and let \( b \) be a continuous function on some interval \( I \). We consider the equation
\[
y' + ay = b(x),
\]
and try to solve it using the same method as in Sec. 5. If \( \phi \) is a solution of (6.1), then
\[
e^{as}(\phi' + a\phi) = e^{as}b,
\]
or
\[
(e^{as}\phi)' = e^{as}b.
\]
Let \( B \) be a function such that \( B'(x) = e^{as}b(x) \), for example,
\[
B(x) = \int_{x_0}^x e^{as}b(t) \, dt,
\]
where \( x_0 \) is some fixed point in \( I \). Since \( e^{as}\phi \) is another function whose derivative is \( e^{as}b \), it follows that
\[
e^{as}\phi(x) = B(x) + c
\]
for some constant \( c \). Therefore
\[
\phi(x) = e^{-as}B(x) + ce^{-as}.
\]
It is easy to see that our steps can be retraced to prove that if \( \phi \) is defined by (6.2), where \( c \) is any constant, then \( \phi \) is a solution of (6.1).

**Exercises**

1. Find all solutions of the following equations:
   (a) \( y' - 2y = 1 \)
   (b) \( y' + y = e^x \)
   (c) \( y' - 2y = x^2 + x \)
   (d) \( 3y' + y = 2e^{-x} \)
   (e) \( y' + 3y = e^x \)

2. Let \( \phi \) be the solution of \( y' + iy = x \) such that \( \phi(0) = 2 \). Find \( \phi(x) \).

3. Consider the equation
   \[ Ly' + Ry = E, \]
   where \( L, R, E \) are positive constants.
   (a) Solve this equation.
   (b) Find the solution \( \phi \) satisfying \( \phi(0) = I_0 \), where \( I_0 \) is a given positive constant.
   (c) Sketch a graph of the solution given in (b) for the case \( I_0 > E/R \).
   (d) Show that every solution tends to \( E/R \) as \( x \to \infty \).

4. Consider the equation
   \[ Ly' + Ry = E \sin \omega x, \]
   where \( L, R, E, \omega \) are positive constants.
   (a) Compute the solution \( \phi \) satisfying \( \phi(0) = 0 \).
   (b) Show that this solution may be written in the form
   \[
   \phi(x) = \frac{EoL}{R^2 + \omega^2 L^2} e^{-(R/L)x} + \frac{E}{\sqrt{R^2 + \omega^2 L^2}} \sin (\omega x - \alpha),
   \]
   where \( \alpha \) is the angle satisfying
   \[
   \cos \alpha = \frac{R}{\sqrt{R^2 + \omega^2 L^2}}, \quad \sin \alpha = \frac{\omega L}{\sqrt{R^2 + \omega^2 L^2}}.
   \]
   (c) Sketch the graph of the solution given in (b).
5. Consider the equation

\[ Ly' + Ry = E e^{i\omega x}, \]

where \( L, R, E, \omega \) are positive constants.

(a) Compute the solution \( \phi \) which satisfies \( \phi(0) = 0 \).

(b) Using the differential equation show that \( \phi_1 = \text{Re} \phi \) satisfies

\[ Ly' + Ry = E \cos \omega x. \]

Compute \( \phi_1 \).

(c) Using the differential equation show that \( \phi_2 = \text{Im} \phi \) satisfies

\[ Ly' + Ry = E \sin \omega x. \]

Compute \( \phi_2 \).

6. Let \( \phi \) satisfy the equation

\[ y' + ay = b_1(x), \]

and let \( \psi \) satisfy the equation

\[ y' + ay = b_2(x), \]

where \( b_1, b_2 \) are defined on the same interval \( I \), and \( a \) is a constant.

(a) Show that \( \chi = \phi + \psi \) satisfies

\[ y' + ay = b_1(x) + b_2(x) \]

on \( I \).

(b) Apply the result of (a) to find the solution of

\[ y' + y = \sin x + 3 \cos 2x \]

whose graph passes through the origin.

7. Consider the equation

\[ y' + ay = b(x), \]

where \( a \) is a constant, and \( b \) is a continuous function on \( 0 \leq x < \infty \), satisfying there \( |b(x)| \leq k \), where \( k \) is some positive number.

(a) Find the solution \( \phi \) satisfying \( \phi(0) = 0 \).

(b) If \( \text{Re} \ a \neq 0 \), show that this solution satisfies

\[ |\phi(x)| \leq \frac{k}{\text{Re} \ a} [1 - e^{-(\text{Re} \ a)x}]. \]

(c) Show that the right side of the inequality in (b) is the solution of

\[ y' + (\text{Re} \ a)y = k, \quad (\text{Re} \ a \neq 0), \]

whose graph passes through the origin.

8. Let \( a \) be a constant, and let \( b_1, b_2 \) be two continuous functions on \( 0 \leq x < \infty \) such that

\[ |b_1(x) - b_2(x)| \leq k, \quad (0 \leq x < \infty), \]

for some constant \( k > 0 \). Let \( \phi \) be a solution of \( y' + ay = b_1(x) \), and \( \psi \) a solution of \( y' + ay = b_2(x) \). Assume that \( \phi(0) = \psi(0) \). Show that

\[ |\phi(x) - \psi(x)| \leq \frac{k}{\text{Re} \ a} [1 - e^{-(\text{Re} \ a)x}] \]

for \( 0 \leq x < \infty \).

(Note: If \( b_2 \) approximates \( b_1 \) with an error at most \( k \), in the sense of (*), then (**) gives an estimate for the difference between the solutions. If \( k \) is small \( \psi \) will be close to \( \phi \).)

9. Consider the equation \( y' + ay = b(x) \), where \( a \) is a constant such that \( \text{Re} \ a > 0 \), and \( b \) is a continuous function on \( 0 \leq x < \infty \) which tends to the constant \( \beta \) as \( x \to \infty \). Prove that every solution of this equation tends to \( \beta/a \) as \( x \to \infty \).

7. The general linear equation of the first order

We now consider the equation

\[ y' + a(x)y = b(x), \quad (7.1) \]

where \( a \) and \( b \) are continuous functions on some interval \( I \). If we are given an equation

\[ \alpha(x)y' + \beta(x)y = \gamma(x) \]

and \( \alpha(x) \neq 0 \) on \( I \), we may divide by \( \alpha(x) \) to obtain an equation of the form (7.1). The points where \( \alpha(x) = 0 \), called singular points, are frequently troublesome. We postpone a discussion of these difficulties until later; see Chap. 4.

We try to solve (7.1) in the same way we solved the ease when \( a \) was constant. Suppose \( \phi \) is a solution of (7.1). We try to find a function \( u \) such that

\[ u(\phi' + a\phi) = (u\phi)' \]

If \( A \) is a function whose derivative is \( a \), for example

\[ A(x) = \int_{x_0}^{x} a(t) \ dt, \]

where \( x_0 \) is a fixed point in \( I \), then such a function \( u \) is given by \( u = e^{A} \) since

\[ (e^{A}\phi)' = e^{A}\phi' + ae^{A}\phi = e^{A}(\phi' + a\phi). \]
Therefore \( \phi' + a\phi = b \) if and only if
\[
(e^a\phi)' = e^a b,
\]
and this is valid if and only if
\[
e^a\phi = B + c,
\]
where \( c \) is a constant, and \( B \) is a function whose derivative is \( e^a b \). For example we can choose \( B \) to be given by
\[
B(x) = \int_{x_0}^{x} e^{A(t)b(t)} dt.
\]
Now (7.2) holds if and only if
\[
\phi(x) = e^{-A(x)}B(x) + ce^{-A(x)}.
\]
We have thus shown that every solution of (7.1) has the form (7.3), and conversely, if \( c \) is any constant, the function \( \phi \) defined by (7.3) is a solution of (7.1).

We remark that the function \( \psi = e^{-A}B \) is a particular solution of (7.1) (the case \( c = 0 \)), and that \( \phi_1 = e^{-A} \) is a solution of the homogeneous equation \( y' + a(x)y = 0 \).

**Theorem 3.** Suppose \( a \) and \( b \) are continuous functions on an interval \( I \). Let \( A \) be a function such that \( A' = a \). Then the function \( \psi \) given by
\[
\psi(x) = e^{-A(x)}\int_{x_0}^{x} e^{A(t)b(t)} dt,
\]
where \( x_0 \) is in \( I \), is a solution of the equation
\[
y' + a(x)y = b(x)
\]
on \( I \). The function \( \phi_1 \) given by
\[
\phi_1(x) = e^{-A(x)}
\]
is a solution of the homogeneous equation
\[
y' + a(x)y = 0.
\]
If \( c \) is any constant, \( \phi = \psi + c\phi_1 \) is a solution of (7.1), and every solution of (7.1) has this form.

In solving a particular linear equation a person with a good memory could remember (7.3), but it is probably easier to remember that multiplication of \( \phi' + a\phi = b \) by \( e^a \) yields \( (e^a\phi)' = e^a b \), which can be immediately integrated to give (7.3). As an example consider the equation
\[
y' + (\cos x)y = \sin x \cos x.
\]

**Exercises**

1. Find all solutions of the following equations:
   (a) \( y' + 2xy = x \)
   (b) \( xy'' + y = 3x^3 - 1 \) (for \( x > 0 \))
   (c) \( y'' + e^y = e^x \)
   (d) \( y' - (\tan x)y = e^{\sin x} \) (for \( 0 < x < \pi/2 \))
   (e) \( y'' + 2xy = xe^{-x^2} \)

2. Consider the equation \( y' + (\cos x)y = e^{-\sin x} \).
   (a) Find the solution \( \phi \) which satisfies \( \phi(x) = \pi \).
   (b) Show that any solution \( \phi \) has the property that
   \[
   \phi(xk) - \phi(0) = \pi k,
   \]
   where \( k \) is any integer.

3. Consider the equation \( xy' + 2xy = 1 \) on \( 0 < x < \infty \).
   (a) Show that every solution tends to zero as \( x \to \infty \).
   (b) Find that solution \( \phi \) which satisfies \( \phi(2) = 2\phi(1) \).

4. Consider the homogeneous equation
   \[
y' + a(x)y = 0,
   \]
   where \( a \) is continuous on an interval \( I \).
   (a) Show that the function \( \phi \) given by \( \phi(x) = 0 \) for all \( x \) in \( I \) (the identically zero function) satisfies this equation. This solution is called the trivial solution.
   (b) If \( \phi \) is any solution of (\( *) \), and \( \phi(x_0) = 0 \) for some \( x_0 \) in \( I \), show that \( \phi \) is the trivial solution.
   (c) If \( \phi, \psi \) are two solutions of (\( *) \) satisfying \( \phi(x_0) = \psi(x_0) \) for some \( x_0 \) in \( I \), show that \( \phi(x) = \psi(x) \) for all \( x \) in \( I \).
   (d) If \( \phi \) is not the trivial solution, and \( \psi \) is any other solution, show that there is a constant \( c \) such that \( \psi = \phi \), that is, \( \psi(x) = \phi(x) \) for all \( x \) in \( I \).
5. The equation
   \[ y' + \alpha(x)y = \beta(x)y^k, \quad (k \text{ constant}), \]
is called Bernoulli's equation.
   (a) Show that the formal substitution \( z = y^{1-k} \) transforms this into the linear equation
   \[ z' + (1 - k)\alpha(x)z = (1 - k)\beta(x). \]
   (b) Find all solutions of \( y' - 2xy = x^2 \).

6. Consider the homogeneous equation \( y' + a(x)y = 0 \), where \( a \) is a continuous function on \( -\infty < x < \infty \) which is periodic with period \( \xi > 0 \), that is, \( a(x + \xi) = a(x) \) for all \( x \).
   (a) Let \( \phi \) be a non-trivial solution, and let \( \psi(x) = \phi(x + \xi) \). Show that \( \psi \) is a solution.
   (b) Show that there is a constant \( c \) such that \( \phi(x + \xi) = c \phi(x) \) for all \( x \).
   (Hint: Ex. 4 (d)). Show that
   \[ c = \exp \left( -\int_0^\xi a(t) \, dt \right). \]
   (Note: \( \exp u \) is an alternate notation for \( e^u \).)
   (c) What condition must \( a \) satisfy in order that there exist a non-trivial solution of period \( \xi \); of period \( 2\xi \)? If \( a \) is real-valued, what is the condition?
   (d) If \( a \) is a constant, what must this constant be in order that a non-trivial solution of period \( 2\xi \) exist?

7. Consider the non-homogeneous equation \( y' + a(x)y = b(x) \), where \( a, b \) are continuous real-valued functions on \( -\infty < x < \infty \) which are of period \( \xi > 0 \), and \( b \) is not identically zero.
   (a) Show that a solution \( \phi \) is periodic of period \( \xi \) if, and only if, \( \phi(0) = \phi(\xi) \).
   (b) Show that there exists a unique solution of period \( \xi \) if there is no non-trivial solution of the homogeneous equation of period \( \xi \).
   (c) Suppose there is a non-trivial periodic solution of the homogeneous equation of period \( \xi \). Show that there are periodic solutions of period \( \xi \) of the non-homogeneous equation if, and only if,
   \[ \int_0^\xi e^{\lambda t} b(t) \, dt = 0, \]
   where \( A(t) = \int_0^t a(s) \, ds \).
   (d) Find solutions of period \( 2\pi \) for the equations:
   (i) \( y' + 3y = \cos x \)
   (ii) \( y' + (\cos 2x)y = \sin 2x \)

8. Find all solutions of the equation
   \[ y' + 2y = b(x), \quad (-\infty < x < \infty), \]
where \( b(x) = 1 - |x| \) for \( |x| \leq 1 \), and \( b(x) = 0 \) for \( |x| > 1 \).

9. The formula
   \[ \psi(x) = e^{-\int_0^x a(t) \, dt} \int_0^x e^{\int_0^y b(t) \, dt} \, dy \]
for a solution \( \psi \) of the equation
   \[ y' + a(x)y = b(x) \]
makes sense for some functions \( b \) which are not continuous. It is sometimes convenient to consider such \( b \), and this \( \psi \) is called a solution even in this case. Of course \( \psi \) satisfies the differential equation at the continuity points of \( b \). Find a solution of the equation
   \[ y' + ay = b(x), \quad (a \text{ constant}), \]
where \( b(x) = 1 \) for \( 0 \leq x \leq \xi \), and \( b(x) = 0 \) for \( x > \xi \). Here \( \xi \) is some positive constant.

10. Suppose \( \phi \) is a function with a continuous derivative on \( 0 \leq x \leq 1 \) satisfying there \( \phi'(x) - 2\phi(x) \leq 1 \), and \( \phi(0) = 1 \). Show that
   \[ \phi(x) \leq \frac{3}{2} e^x - \frac{1}{2}. \]

11. Let \( \phi, \psi \) be solutions of \( y' + a(x)y = b(x) \) on an interval \( I \) containing \( x_0 \). Show that for \( x \in I \),
   \[ \psi(x) - \phi(x) = (\psi(x_0) - \phi(x_0)) \exp \left[ -\int_{x_0}^x a(t) \, dt \right], \]
and consequently that
   \[ |\psi(x) - \phi(x)| = |\psi(x_0) - \phi(x_0)| \exp \left[ -\int_{x_0}^x \Re a(t) \, dt \right]. \]

12. Consider the boundary value problem
   \[ iy' = iy, \quad y(1) = e^{ix} y(0), \]
where \( \alpha \) is a fixed real number, and \( i \) is a complex number.
   (a) Show that this problem has a non-trivial solution if, and only if,
   \[ l = \lambda_k = 2\pi k - \alpha, \]
where \( k = 0, \pm 1, \pm 2, \ldots \).
   (b) Compute a solution \( \phi_k \) of the problem for \( l = \lambda_k \) which satisfies
   \[ \int_0^1 |\phi_k(x)|^2 \, dx = 1. \]
   (c) If \( \phi_k, \phi_k \) are the solutions determined in (b) for \( l = \lambda_k, l = \lambda_k \) respectively, show that
   \[ \int_0^1 \phi_k(x)\bar{\phi_k}(x) \, dx = 0 \]
if \( \lambda_k \neq \lambda_k \).
(d) If $f$ is a function having the form
$$f = A_1 \phi_1 + \cdots + A_n \phi_n,$$
where the $\phi_k$ are as in (b), and the $A_k$ are constants, show that
$$A_k = \int_0^1 f(x) \phi_k(x) \, dx.$$
(Hint: Use (b) and (c).)

13. Let $f$ be any continuous function on $0 \leq x \leq 1$, and consider the problem
$$(iy' - iy = f(x), \quad y(1) = e^{\alpha y(0)},)$$
where $\alpha$ is real, and $i$ is a complex number not equal to any of the $\lambda_k$ in Ex. 12, (a). Find a solution $y$ of this problem, and show that it can be expressed in the form
$$y(x) = \int_0^1 g(x, y) dy,$$
where $g$ has a discontinuity at $y = x$.

14. (a) Find the solution $\phi$ of the linear equation
$$y' = 1 + y$$
satisfying $\phi(0) = 0$. Observe that this solution exists for all real $z$.
(b) Find the real-valued solution $\psi$ of the nonlinear equation
$$y' = 1 + y^2$$
satisfying $\psi(0) = 0$. Observe that this solution exists only for $-(\pi/2) < x < (\pi/2)$. (Hint: For any $t$ for which such a $\psi$ exists we must have
$$\frac{\psi'(t)}{1 + [\psi(t)]^2} = (\tan^{-1} \psi'(t)) = 1.$$ Integrating from 0 to $x$ we obtain $\tan^{-1} \psi(x) = x$, or $\psi(x) = \tan x$. Check that this $\psi$ is the solution desired.)

(Note: This illustrates one of the differences between linear and nonlinear equations. General techniques for solving equations such as in (b) will be considered in Chap. 5.)

CHAPTER 2

Linear Equations with Constant Coefficients

1. Introduction

A linear differential equation of order $n$ with constant coefficients is an equation of the form
$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y = b(x),$$
where $a_0 \neq 0$, $a_1, \ldots, a_n$ are complex constants, and $b$ is some complex-valued function on an interval $I$. By dividing by $a_n$ we can arrive at an equation of the same form with $a_n$ replaced by 1. Therefore we can always assume $a_n = 1$, and our equation becomes
$$y^{(n)} + a_{n-1} y^{(n-1)} + a_1 y = b(x). \quad (1.1)$$

It will be convenient to denote the differential expression on the left of the equality (1.1) by $L(y)$. Thus
$$L(y) = y^{(n)} + a_{n-1} y^{(n-1)} + a_1 y,$$
and the equation (1.1) becomes simply $L(y) = b(x)$. If $b(x) = 0$ for all $x$ in $I$ the corresponding equation $L(y) = 0$ is called a homogeneous equation, whereas if $b(x) \neq 0$ for some $x$ in $I$, $L(y) = b(x)$ is called a nonhomogeneous equation.

We give a meaning to $L$ itself as a differential operator which operates on functions which have $n$ derivatives on $I$, and transforms such a function $\phi$ into a function $L(\phi)$ whose value at $x$ is given by
$$L(\phi)(x) = \phi^{(n)}(x) + a_{n-1} \phi^{(n-1)}(x) + \cdots + a_1 \phi(x).$$

Thus
$$L(\phi) = \phi^{(n)} + a_{n-1} \phi^{(n-1)} + \cdots + a_1 \phi.$$
Chap. 0, Sec. 4

1. (a) \(-3\), multiplicity 1; 2, multiplicity 1
   (b) \(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\), multiplicity 1; \(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\), multiplicity 1
   (c) 2, multiplicity 2; 1, multiplicity 1
   (d) 1, multiplicity 2; \(i\), multiplicity 1
   (e) \(3^{1/4}\), \(-3^{1/4}\), \(3^{1/4}i\), \(-3^{1/4}i\), all multiplicity 1

6. (a) \(i\) has multiplicity 3
    (b) \(-1 + i\), \(-1 - i\)

Chap. 0, Sec. 5

1. \(1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i\)

2. \(\frac{\sqrt{2}}{2} (1 + i), -\frac{\sqrt{2}}{2} (1 + i)\)

3. (a) \((24)^{1/4} \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right), - (24)^{1/4} \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\)
   (b) \(2\sqrt{2} \cos \frac{3\pi}{8} + i \sin \frac{3\pi}{8}\),
   \(2\sqrt{2} \cos \frac{7\pi}{8} + i \sin \frac{7\pi}{8}\),
   \(2\sqrt{2} \cos \frac{11\pi}{8} + i \sin \frac{11\pi}{8}\),
   \(2\sqrt{2} \cos \frac{15\pi}{8} + i \sin \frac{15\pi}{8}\)
   \(\sqrt{2}i, \sqrt{2}i, -\sqrt{2}i, -\sqrt{2}i\)
   (d) \(\cos \left(\frac{2\pi k}{100}\right) + i \sin \left(\frac{2\pi k}{100}\right), k = 0, 1, 2, \ldots, 99\)

6. (b) \(\frac{ae^a \cos b + be^a \sin b - a}{a^2 + b^2}\)

Chap. 1, Sec. 3

1. (a) \(\phi(x) = \frac{e^x}{3} - \cos x + c\), \(c\) any constant

   (b) \(\phi(x) = x^2 + \frac{x^3}{6} + c_1 x + c_2\), \(c_1, c_2\) any constants

   (c) \(\phi(x) = c_1 x^{k-1} + c_2 x^{k-2} + \cdots + c_k\), \(c_1, c_2, \ldots, c_k\) any constants

   (d) \(\phi(x) = \frac{x^3}{60} + c_1 x^3 + c_2 x^2 + c_3 x^2 + c_4, c_1, c_2, c_3, c_4\) any constants

3. (b) \(\phi(x) = \left(\frac{2}{5}\right) \left[1 + 4e^{(3-x)}\right]\)

   (c) \(\phi(x) = \left(\frac{2}{5}\right) \left[1 - \frac{2}{3 - e^{-4x}}\right]\)

4. (a) \(\phi(x) = \frac{x^3}{2} + \frac{x^3}{2} + c_2 x + c_3, (0 \leq x \leq 1), c_1, c_3\) any constants
(b) \( \phi(x) = \frac{x^3}{2} + \frac{x^2}{2} + 2x + 1 \)

(c) \( \phi(x) = \frac{x^3}{2} + \frac{x^2}{2} + 2x \)

5. (d) \( |\phi(x)| = |\phi(0)| \) for all \( x \)

Chap. 1, Sec. 6

1. (a) \( \phi(x) = -\frac{1}{2} + ce^{2x}, \ c \ any \ constant \)
   (b) \( \phi(x) = \frac{1}{2} e^x + ce^{-x}, \ c \ any \ constant \)
   (c) \( \phi(x) = -\frac{1}{2}(2x^2 + 2x + 1) + ce^{2x}, \ c \ any \ constant \)
   (d) \( \phi(x) = -e^x + ce^{-x}, \ c \ any \ constant \)
   (e) \( \phi(x) = \left( \frac{3 - i}{10} \right)e^x + ce^{-x}, \ c \ any \ constant \)

2. \( \phi(x) = -ix \)

3. (a) \( \phi(x) = \frac{E}{R} + ce^{-RzLx}, \ c \ any \ constant \)
   (b) \( \phi(x) = \frac{E}{R} (1 - e^{-RzLx}) \)

4. (a) \( \phi(x) = \frac{E}{R^2 + \omega^2 L^2}(\omega L e^{-RzLx} + R \sin \omega x - \omega L \cos \omega x) \)
   (b) \( \phi(x) = \frac{E}{R^2 + \omega^2 L^2}(e^{\omega x} - e^{-RzLx}) \)

5. (a) \( \phi_1(x) = \frac{E}{R^2 + \omega^2 L^2}(-Re^{-RzLx} + R \cos \omega x + \omega L \sin \omega x) \)
   (b) \( \phi_2(x) = \frac{E}{R^2 + \omega^2 L^2}(\omega L e^{-RzLx} + R \sin \omega x - \omega L \cos \omega x) \)
   (c) \( \phi_2(x) = \frac{E}{R^2 + \omega^2 L^2}(\omega L e^{-RzLx} + R \sin \omega x - \omega L \cos \omega x) \)

6. (b) \( \phi(x) = \frac{1}{2} \sin x + \frac{1}{2} \cos x + \frac{3}{2} \cos 2x + \frac{3}{2} \sin 2x - \frac{x}{3} e^{-x} \)

7. (a) \( \phi(x) = e^{-ax} \int_0^x e^{bt} b(t) \, dt \)

Given any \( \epsilon > 0 \) there is an \( x_0 > 0 \) such that \( |b(x) - \beta| < \epsilon \) for \( x \geq x_0 \).
Write any solution \( \phi \) as
\[ \phi(x) = e^{-ax} \phi(x_0) + \int_{x_0}^x e^{-a(t-x)} b(t) \, dt \]

Chap. 1, Sec. 7

1. (a) \( \phi(x) = \frac{1}{2} + ce^{-ax}, \ c \ any \ constant \)
   (b) \( \phi(x) = \frac{3}{4} x^2 - 1 + \frac{c}{x}, \ (x > 0), \ c \ any \ constant \)
   (c) \( \phi(x) = 3 + c \frac{e^{x}}{x}, \ c \ any \ constant \)
   (d) \( \phi(x) = (sec x)e^{sec x} + c \ sec x, \ c \ any \ constant \)
   (e) \( \phi(x) = \frac{x^2}{2} \ exp (x^2) + c \ exp (-x^2), \ c \ any \ constant \)

2. (a) \( \phi(x) = xe^{-x} \sin x \)

3. (b) \( \phi(x) = \frac{1}{x} - \frac{6}{7x^2}, \ (x > 0) \)

5. (b) \( \phi(x) = (\frac{1}{2} + ce^{-ax})^{-1}; \ also \( \phi(x) = 0 \)

6. (c) \( \int_0^t a(t) \, dt = 2\pi ik, \ (k = 0, \pm 1, \pm 2, \cdots) \)
   \( \int_0^t a(t) \, dt = \pi ik, \ (k = 0, \pm 1, \pm 2, \cdots) \)
   \( \int_0^t a(t) \, dt = 0 \)

7. (d) (i) \( \phi(x) = \frac{1}{2} (3 \cos x + \sin x) \)
   (ii) \( \phi(x) = ce^{-x} \sin x + 2 \sin (x - 1), \ c \ any \ constant \)

8. \( \phi(x) = c - \frac{x}{e^x - 1} e^{-x} \), \( (-\infty < x \leq -1) \)
   \( \phi(x) = c - \frac{1}{e^x - 1} e^{-x} e^{x} + \frac{1}{2}(1 + 2x), \ (-1 < x \leq 0) \)
   \( \phi(x) = c - \frac{1}{e^x - 1} e^{-x} e^{x} + \frac{1}{3}(3 - 2x), \ (0 < x \leq 1) \)
   \( \phi(x) = c + \frac{1}{e^x - 1} e^{-x} e^{x}, \ (1 < x < \infty), c = 0 \)

9. (a) \( \psi(x) = x, \ (0 \leq x \leq \xi), \ and \( \psi(x) = \xi, \ (x > \xi) \)
   (b) \( \psi(x) = \phi(x) - \phi(0), \ (0 \leq x \leq \xi) \), and \( \psi(x) = (\phi(x) - \phi(0)) - e^{-ax}, \ (0 \leq x \leq \xi) \), and \( \psi(x) = (\phi(x) - \phi(0)) e^{-ax} - 1, \ (x > \xi) \)

12. (b) \( \psi_1(x) = \exp \left[ -\frac{i}{2}(3 \pi k - a) \right] \)

13. \( g(x, y) = \frac{1}{1 - e^{(x-y) - (x+y)}} \), \( (0 \leq y \leq x) \)

14. (a) \( \phi(x) = e^x - 1 \)