1. If $a = 2^4 13^2 19$ and $b = 2^3 5^2 13$ then find the prime factorization of
   (a) (a, b)
   (b) $[a, b]$
   (c) (a^2, b^3)
2. Prove that any whole number amount greater than 23 cents could be made up using an unlimited supply of 5 cent and 7 cent coupons.
3. (a) Evaluate $\phi(3000)$.
   (b) Find the remainder when 11^{2402} is divided by 3000.
4. Use induction to prove that $6^n \equiv 5n + 1 \pmod{25}$ for all positive integers n.
5. Use induction to prove that $2^n \mid (2n)!$
6. Give a proof that there are infinitely many primes.
7. Find all right-angled triangles with relatively prime integer sides and base of given length:
   (a) 28
   (b) 55
8. (a) Find the prime factorization of 600.
   (b) $\tau(n)$ is the number of positive divisors of n. Evaluate $\tau(600)$.
   (c) $\sigma(n)$ is the sum of the positive divisors of n. Evaluate $\sigma(600)$.
   (d) $\phi(n)$ is the Euler phi function. Evaluate $\phi(600)$.
   (e) $\mu(n)$ is the Möbius function. Evaluate $\mu(600)$.
9. Give a non-trivial factor of $2^{55} - 1$. (Bonus points for two.)
10. Prove that if $a \mid b$ and $a \mid c$ then $a^2 \mid 7bc$.
11. Use congruences to prove that $x^2 - 5y^2 = 3$ has no integer solutions.
12. (a) If $F(n) = \sum_{d|n} \sigma(d)$ then evaluate $F(175)$.
    (b) Evaluate $\sigma(22,491)$. (Hint: $22,491 = 27 \cdot 49 \cdot 17$.)
13. Suppose $g(n)$ is a multiplicative function satisfying $\tau(n)^2 = \sum_{d|n} g(d)$.
    (a) Use the Möbius inversion formula to give a formula for $g(n)$.
    (b) Evaluate $g(5^3)$.
    (c) Evaluate $g(700)$.
14. (a) What can you say about the prime factorization of n if $\tau(n) = 8$?
    (b) What is the smallest n with $\tau(n) = 8$.
    (c) Find three n with $\phi(n) = 16$.
15. (a) Suppose that $d = \text{ord}_m\, a$. Prove that if $a^n \equiv 1 \pmod{m}$ then $d \mid n$.

(b) Find (with justification) $\text{ord}_m b$ if $b^8 \equiv -1 \pmod{m}$ with $m \geq 2$.

16. (a) What is the order of 3 modulo 23?

    (b) If the order of b modulo m is 15, what is the order of b^6 modulo m.

17. Use the Chinese Remainder Theorem to solve the simultaneous congruences:

$$\begin{aligned} x &\equiv 3 \pmod{5} \\ x &\equiv 2 \pmod{7} \\ x &\equiv 1 \pmod{6} \end{aligned}$$

18. (a) Use the Euclidean algorithm to compute the greatest common divisor $(2517, 2370)$.

    (b) Find all integer solutions to the equation $2517x - 2370y = 69$, or explain why there are none.

    (c) Solve the linear congruence $2370x \equiv 69 \pmod{2517}$ or explain why there are no solutions.

19. (a) For which odd primes p does the Legendre symbol $\left(\frac{2}{p}\right) = 1$?

    (b) For which distinct odd primes p, q does the Legendre symbol satisfy $\left(\frac{p}{q}\right) = -\left(\frac{q}{p}\right)$?

    (c) Evaluate the Legendre symbol $\left(\frac{431}{1097}\right)$.

20. Find a complete solution to the congruence $x^2 - 5x + 6 \equiv 0 \pmod{187}$. (Note that $187 = 11 \cdot 17$.)

21. Solve $x^2 + x + 2 \equiv 0 \pmod{121}$.

22. Determine if each of the following congruences have a solution.

    (a) $x^2 \equiv 15 \pmod{41}$.

    (b) $x^2 + 5x + 7 \equiv 0 \pmod{97}$.

    (c) $3x^2 + 4x + 5 \equiv 0 \pmod{51}$.

23. Circle True (T) or False (F). Reasons are not required.

T	F	(a) If $15 \mid a^2$ then $15 \mid a$.
T	F	(b) If $x^2 \equiv 1 \pmod{35}$ then $x \equiv \pm 1 \pmod{35}$.
T	F	(c) $\{21, -3, 13, -15, -4\}$ is a complete residue system modulo 5.
T	F	(d) $7^{753} \equiv 2 \pmod{11}$.
T	F	(e) $\{1, 3, -3, 9\}$ is a reduced residue system modulo 10.
T	F	(f) The Fibonacci numbers satisfy $f_{2n+3} - f_{2n+2} = f_{2n+1}$.
T	F	(g) $\underbrace{72727272727272727272}_{10\text{ times}} \equiv 6 \pmod{11}$.
T	F	(h) If p is an odd prime then $2^p \equiv 2 \pmod{p}$.
T	F	(i) The composition $\tau(\tau(n))$ is a multiplicative function.
T	F	(j) If p is prime then $\phi(pm) = \phi(m)$.
T	F	(k) $\sum_{d \mid n} \phi(d) = n$.