Instructions. Answer each of the questions on your own paper (except for problem 1, which should be answered on this paper), and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.

1. [**10 Points**] Fill in the blanks to complete the definition of group:

   A *group* is a set G together with a binary operation $*$ and a special element $e \in G$, called the ______________, such that

   (a) for every x, y, $z \in G$,
   $$x * (y * z) = \underline{\hspace{3cm}};$$

   (b) for all $x \in G$,
   $$x * e = \underline{\hspace{1.5cm}} = e * x;$$

   (c) for every $x \in G$, there is $x' \in G$ with
   $$x * x' = \underline{\hspace{1.5cm}} = x' * x.$$

   The element x' is called the ______________ of x.

2. [**20 Points**] Let G be a group and let $a \in G$.

   (a) Complete the definition of what it means for a to have finite order $o(a) = n$: *If n is a positive integer, then the element $a \in G$ has order n provided*

   (b) Let $a \in G$ be an element of order $n = 3m$ and let $b = a^3$. What is the order of b? Prove that your answer is correct directly from the definition of order given in part (a).

3. [**20 Points**] Let $G = \mathbb{Z}_{13}^*$.

   (a) What is the order of G?

   (b) If $a \in G$, what are the possibilities for the order of a?

   (c) What is the order of the element $[2] \in G$?

   (d) Show that G is a cyclic group.

4. [**16 Points**]

   (a) State Lagrange's theorem.

   (b) Suppose that H and K are subgroups of G and assume that the following data are given: $|H| = 9$, $|K| = 12$, $|G| < 100$. What are the possible values of $|G|$?

5. [**16 Points**]

   (a) Give an example of two groups of order 6 which are not isomorphic. Explain why your examples are not isomorphic.

   (b) Give an example of two groups of order 4 which are not isomorphic. Explain why your examples are not isomorphic.

6. **[18 Points]** Let $H = \left\{ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},\ B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$.

   (a) The following table gives the multiplications between elements of H:

$\cdot$	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	A	I
C	C	B	I	A

   Explain why the above multiplication table shows that H is a subgroup of the group $\mathrm{GL}_2(\mathbb{R})$ of invertible 2×2 matrices with group operation matrix multiplication.

   (b) Verify that $K = \{I, B\}$ is *not* a subgroup of H.

   (c) The set K is not a subgroup, but there is a subgroup of H consisting of 2 elements. Which two elements of H form a subgroup?