Instructions. Answer each of the questions on your own paper and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.

1. [**24 Points**] Suppose that Q is the group defined by the following multiplication table. We are denoting the identity of Q by the symbol 1, and we have also outlined the rows and columns beginning with the element a, to facilitate the reading of the multiplication by a. This may be useful in some of the questions that follow.

$\cdot$	1	a	b	c	d	e	f	g
1	1	a	b	c	d	e	f	g
a	a	1	c	b	e	d	g	f
b	b	c	a	1	f	g	e	d
c	c	b	1	a	g	f	d	e
d	d	e	g	f	a	1	b	c
e	e	d	f	g	1	a	c	b
f	f	g	d	e	c	b	a	1
g	g	f	e	d	b	c	1	a

   (a) Verify that $H = \{1, a\}$ is a subgroup of Q.

   (b) List all of the *distinct* left cosets of H in Q.

   (c) Is H a normal subgroup of Q? *Hint: Observe from the multiplication table that $ax = xa$ for all $x \in Q$.*

   (d) Write the multiplication table for the factor group Q/H.

   (e) Is Q/H a cyclic group? Explain.

2. [**12 Points**] For each of the following statements, fill in the appropriate hypotheses or conclusions to ensure that the statement is a theorem proved in class.

   (a) If F is a field, the space of congruence classes $F[x]/\langle p(x)\rangle$ is a field if and only if $p(x)$ is an ____________________ polynomial over F.

   (b) A polynomial $p(x)$ of degree ____________________ is irreducible over the field F if and only if $p(x)$ has no roots in F.

   (c) **(Remainder Theorem).** Let $f(x) \in F[x]$ be a nonzero polynomial, and let $c \in F$. Then there exists a polynomial $q(x) \in F[x]$ such that

$$f(x) = q(x)(x - c) + \boxed{}.$$

3. [**20 Points**] Let $f(x) = x^2 - 1$ and let $g(x) = x^3 + 1$ be polynomials in $\mathbb{Q}[x]$.

   (a) Use Euclid's Algorithm to find $d(x) = \gcd(f(x), g(x))$.

   (b) Express $d(x)$ in the form $d(x) = a(x)f(x) + b(x)g(x)$.

4. **[20 Points]** Determine whether each of the following polynomials is irreducible over the given field. Justify your answers. If the polynomial is reducible, find a factorization into irreducible polynomials.

   (a) $x^3 + x^2 + 2x + 2$ over $\mathbb{Z}_3$.

   (b) $3x^5 + 4x^3 + 6$ over $\mathbb{Q}$.

5. **[24 Points]** Let $E = \mathbb{Z}_2[x]/\langle x^3 + x + 1 \rangle$.

   (a) List all of the distinct elements of E.

   (b) Calculate the product $[x^2 + 1][x^2 + x + 1]$ and identify this element on the list produced in part (a).

   (c) Find the multiplicative inverse of $[x + 1]$.