Instructions. Answer each of the questions on your own paper. Be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.

1. [**15 Points**] Solve the congruence $5x \equiv 23 \mod 32$.

2. [**15 Points**] Let $\alpha = (3, 1, 7, 5, 9)$, $\beta = (6, 2, 8, 4)$ and $\sigma = \alpha\beta$ in S_9.

   (a) What are the orders of α, β and σ?

   (b) Find the smallest positive integer m such that $\sigma^m = \alpha$.

   (c) Find *all* of the integers m such that $\sigma^m = \alpha$.

3. [**30 Points**] Let $G = \mathbb{Z}_{16}^* = \{[r] \in \mathbb{Z}_{16} : \gcd(r, 16) = 1\}$. Recall that $\mathbb{Z}_{16}^*$ is a group under the operation of multiplication modulo 16. Let $H = \langle [7] \rangle$ be the cyclic subgroup of G generated by [7] and let $K = \langle [9] \rangle$ be the cyclic subgroup of G generated by [9].

   (a) List the elements of G, the elements of H and the elements of K.

   (b) Are H and K isomorphic groups?

   (c) List all of the distinct cosets of H in G.

   (d) List all of the distinct cosets of K in G.

   (e) Write the multiplication table for G/H.

   (f) Write the multiplication table for G/K.

   (g) By inspecting your multiplication tables determine if the two factor groups G/H and G/K are isomorphic. Be sure to explain, by specific references to the tables, how you arrived at your conclusion.

4. [**30 Points**] Each of the following statements may be either **True** or **False**. Determine which and give a brief reason why.

   (a) If G is a group with 13 elements, then the only subgroups of G are the identity subgroup $\{e\}$ and the entire group G itself.

   (b) If G is a group of order $|G| = 15$ and H is a subgroup with $|H| = 3$, then every coset of H in G has 5 elements.

   (c) If $\sigma \in S_6$, then the order of σ divides 6.

   (d) The polynomial $p(x) = x^5 + 12x^3 + 6x^2 - 24x + 18$ is irreducible in $\mathbb{Q}[x]$.

   (e) There are polynomials $p(x)$ and $q(x) \in \mathbb{Z}_5[x]$ such that

   $$p(x)(x^5 - 2x^3 + 4) + q(x)(x^2 - 1) = 1.$$

   (f) $\mathbb{Z}_{47}$ is a field.

5. [**15 Points**] Let G be a group such that $x^2 = e$ for every $x \in G$. Prove that G is abelian.

6. **[15 Points]** Let $G = \mathbb{Z}_7^*$.

   (a) Show that G is a cyclic group by finding a generator.

   (b) List *all* of the subgroups of G.

7. **[15 Points]** Let $E = \mathbb{Q}[x]/\langle x^2 - 5\rangle$ be the set of congruence classes of polynomials modulo $x^2 - 5$. As usual, we will denote the congruence class of $p(x) \in \mathbb{Q}[x]$ by the symbol $[p(x)]$.

   (a) Explain briefly why $E = \mathbb{Q}[x]/\langle x^2 - 5\rangle$ is a field.

   (b) Find a and b in $\mathbb{Q}$ so that $[x^3 - 2x^2 - 4x + 12] = [a + bx]$.

   (c) Find the multiplicative inverse of $[x^3 - 2x^2 - 4x + 12]$ in E.

8. **[15 Points]** Let $G = \langle a\rangle$ be a cyclic group of order 9 with generator a and let $H = S_3$.

   (a) Complete the following table so as to make the function $f : G \to H$ a homomorphism. (*Hint:* Use the fact that a homomorphism satisfies $f(a^2) = f(a)f(a)$, $f(a^3) = f(a^2)f(a)$, etc.)

g	1	a	a^2	a^3	a^4	a^5	a^6	a^7	a^8
$f(g)$	(1)	(1, 2, 3)							

   (b) Find the kernel of f. (Recall that $\mathrm{Ker}(f) = \{x \in G : f(x) = e\}$, where e is the identity of the group H.)

   (c) Find the image of f. (Recall that $\mathrm{Im}(f) = \{y \in H : y = f(x) \text{ for some } x \in G\}$.)