Instructions. Answer each of the questions on your own paper and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.

Some useful notation: $\mathbb{Z}$ is the group of integers under addition; $\mathbb{Z}_n$ is the group of congruence classes modulo n under addition of congruence classes; $\mathbb{Z}_n^*$ is the group of invertible congruence classes modulo n under multiplication of congruence classes; S_n is the group of permutations of the set $\{1, \ldots, n\}$ under composition of permutations.

1. [**15 Points**] Let $\sigma = (2, 4, 9, 7)(6, 4, 2, 5, 9)(1, 6)(3, 8, 6) \in S_9$.
   (a) Write σ as a product of disjoint cycles.
   (b) What is the order of σ.
   (c) Write σ^{-1} as a product of disjoint cycles.
2. [**18 Points**]
   (a) If G is a group and H is a subset of G, list the conditions that H must satisfy to guarantee that H is a subgroup.
   (b) Decide in each of the following cases whether the given subset is a subgroup of the group S_4. Justify your answer.
      i. $H_1 = \{(1), (1, 3, 4), (1, 4, 3)\}$
      ii. $H_2 = \{(1), (1, 2, 3, 4), (1, 4, 3, 2)\}$
3. [**18 Points**] Let G be a group and let $a \in G$.
   (a) Complete the definition of what it means for a to have finite order $o(a) = n$: *If n is a positive integer, then the element $a \in G$ has order n provided*
   (b) Let $a \in G$ be an element of order 20 and let $b = a^5$. What is the order of b? Prove that your answer is correct directly from the definition of order given in part (a).
4. [**18 Points**] $\mathbb{Z}_7^* = \{1, 2, 3, 4, 5, 6\}$ is the group of invertible congruence classes modulo 7 under multiplication.
   (a) Compute the cyclic subgroup $\langle 2 \rangle$ of $\mathbb{Z}_7^*$ generated by 2.
   (b) Compute the cyclic subgroup $\langle 3 \rangle$ of $\mathbb{Z}_7^*$ generated by 3.
   (c) Show that $\mathbb{Z}_7^*$ is a cyclic group and give a generator of $\mathbb{Z}_7^*$.
5. [**16 Points**]
   (a) State Lagrange's theorem.
   (b) Suppose that a finite group G has a subgroup of order 4 and and another of order 10 and assume that $|G| < 50$. What can you conclude about $|G|$? Justify your answer.

6. **[15 Points]** Assume $H = \{u, v, w, x, y, z\}$ is a group with respect to multiplication and $\varphi : \mathbb{Z}_6 \to H$ is a group isomorphism with

$$\varphi([0]_6) = u, \quad \varphi([1]_6) = v, \quad \varphi([2]_6) = w,$$
$$\varphi([3]_6) = x, \quad \varphi([4]_6) = y, \quad \varphi([5]_6) = z.$$

Replace each of the following products by the appropriate element of H, i.e., either u, v, w, x, y, or z.

(a) Identity of H (b) xw (c) w^{-1} (d) w^5 (e) $zy^{-1}x$

[Extra Credit Problem, 8 Points] Let p and q be distinct primes. Suppose that H is a proper subgroup of $\mathbb{Z}$ (i.e., $H \neq \mathbb{Z}$). Recall that the group operation is addition. Assume that H contains exactly three elements of the following set

$$\{p,\, p+q,\, pq,\, p^q,\, q^p\}\,.$$

Determine which of the following are the three elements in H.

(a) pq, p^q, q^p

(b) $p+q$, pq, p^q

(c) p, $p+q$, pq

(d) p, p^q, q^p

(e) p, pq, p^q