Instructions. Answer each of the questions on your own paper and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.

1. [**20 Points**] Let $G = \langle a \rangle$ be a cyclic group of order 45.

   (a) Compute the order of each of the following elements: (i) a^2 (ii) a^5 (iii) a^{27}

   (b) How many generators of G are there?

   (c) Find all of the subgroups of G and draw the subgroup diagram for G.

2. [**25 Points**]

   (a) Complete the definition of *group homomorphism*: If G and G' are groups, a function $\varphi : G \to G'$ is a group homomorphism if

   $$\boxed{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}\,.$$

   (b) Give the definition of the *kernel* of a group homomorphism.

   (c) In each case determine whether $\varphi : G \to G_1$ is a group homomorphism. Use the definition you provided in part (a) to prove that your answer is correct.

   i. $G = G_1 = \mathbb{Z}_7^*$, $\varphi(a) = a^2$.
   ii. $G = G_1 = S_3$, $\varphi(a) = a^2$.

   (d) For each function φ in part (c) that is a group homomorphism, find the kernel of φ, denoted $\mathrm{Ker}(\varphi)$, and the image $\varphi(G)$.

3. [**25 Points**] Recall that the dihedral group D_4 is defined by generators and relations as

$$\begin{aligned} D_4 &= \langle a, b | a^4 = e, b^2 = e, ba = a^{-1}b \rangle \\ &= \left\{ e, a, a^2, a^3, b, ab, a^2b, a^3b \right\}. \end{aligned}$$

For convenience the multiplication table for D_4 is given here:

$\cdot$	e	a	a^2	a^3	b	ab	a^2b	a^3b
e	e	a	a^2	a^3	b	ab	a^2b	a^3b
a	a	a^2	a^3	e	ab	a^2b	a^3b	b
a^2	a^2	a^3	e	a	a^2b	a^3b	b	ab
a^3	a^3	e	a	a^2	a^3b	b	ab	a^2b
b	b	a^3b	a^2b	ab	e	a^3	a^2	a
ab	ab	b	a^3b	a^2b	a	e	a^3	a^2
a^2b	a^2b	ab	b	a^3b	a^2	a	e	a^3
a^3b	a^3b	a^2b	ab	b	a^3	a^2	a	e

   (a) List all of the *distinct* left cosets of the subgroup $H = \{e, a^2\}$ in D_4.

(b) Verify that H a normal subgroup of D_4? You may assume that H is a subgroup. It is only necessary to verify that H is *normal. Hint: Observe from the multiplication table that* $a^2x = xa^2$ *for all* $x \in D_4$.

(c) Write the multiplication table for the factor group D_4/H.

(d) Is D_4/H a cyclic group? Explain.

4. [**10 Points**] Compute the number of polynomials in $\mathbb{Z}_5[x]$ of degree 4.

5. [**20 Points**] Let $f(x) = x^3 - 1$ and let $g(x) = x^4 + x^3 + 2x^2 + x + 1$ be polynomials in $\mathbb{Z}_5[x]$.

   (a) Use the Remainder Theorem to determine if $x - 2$ divides $g(x)$ in $\mathbb{Z}_5[x]$.

   (b) Use Euclid's Algorithm to find $d(x) = \gcd(f(x),\, g(x))$.

   (c) Express $d(x)$ in the form $d(x) = a(x)f(x) + b(x)g(x)$, for polynomials $a(x)$, $b(x) \in \mathbb{Z}_5[x]$.