Instructions. Answer each of the questions on your own paper, and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.

1. [**16 Points**] Calculate the greatest common divisor $d = \gcd(525, 231)$ by the Euclidean algorithm, and write d in the form $525 \cdot s + 231 \cdot t$ for some integers s and t.

   ▶ **Solution.** Apply the Euclidean Algorithm:

$$\begin{aligned} 525 &= 2 \cdot 231 + 63 \\ 231 &= 3 \cdot 63 + 42 \\ 63 &= 1 \cdot 42 + 21 \\ 42 &= 2 \cdot 21 + 0. \end{aligned}$$

   Thus, $d = \gcd(525, 231) = 21$.

   Reverse the above calculations to write d as a linear combination:

$$\begin{aligned} 21 &= 63 - 42 \\ &= 63 - (231 - 3 \cdot 63) \\ &= 4 \cdot 63 - 231 \\ &= 4(525 - 2 \cdot 231) - 231 \\ &= 4 \cdot 525 - 9 \cdot 231. \end{aligned}$$

   Thus $21 = 4 \cdot 525 - 9 \cdot 231$. ◀

2. [**18 Points**] Use induction to prove that the equation

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

   is valid for all integers $n \geq 1$. That is, the sum of the first n odd integers is equal to n^2.

   ▶ **Solution.** Let $P(n)$ be the statement "$1 + 3 + 5 + \cdots + (2n - 1) = n^2$". Then $P(1)$ is the statement "$1 = 1^2$", which is true. Now assume that $P(k)$ is true. That is , assume that "$1 + 3 + 5 + \cdots + (2k - 1) = k^2$". Then,

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2k - 1) + (2(k + 1) - 1) &= k^2 + 2k + 1 \\ &= (k + 1)^2. \end{aligned}$$

   Therefore, whenever $P(k)$ is true, $P(k + 1)$ is also true. By the induction principle, $P(n)$ is thus true for all $n \geq 1$. ◀

3. [**18 Points**] Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} A$ satisfy $\beta\alpha = 1_A$.

   (a) Show that α is one-to-one.

▶ **Solution.** Suppose that $\alpha(a_1) = \alpha(a_2)$. Then

$$\begin{aligned} a_1 &= 1_A(a_1) \\ &= \beta\alpha(a_1) \\ &= \beta(\alpha(a_1)) \\ &= \beta(\alpha(a_2)) \\ &= \beta\alpha(a_2) \\ &= 1_A(a_2) \\ &= a_2. \end{aligned}$$

Thus, α is one-to-one. ◀

(b) Give an example with $A = \mathbb{N} = B$ such that α does not have an inverse.

▶ **Solution.** Let $\alpha : \mathbb{N} \to \mathbb{N}$ by $\alpha(n) = n+1$, and $\beta : \mathbb{N} \to \mathbb{N}$ by $\beta(n) = \begin{cases} n-1 & \text{if } n \geq 1, \\ 0 & \text{if } n = 0. \end{cases}$
Then $\beta\alpha(n) = \beta(\alpha(n)) = \beta(n+1) = (n+1)-1 = n$, so $\beta\alpha = 1_A$, but α is not surjective since $0 \notin \mathrm{Im}(\alpha)$, so α is not invertible. ◀

4. [**16 Points**] Find all solutions to the system of simultaneous linear congruences:

$$\begin{aligned} x &\equiv 9 \pmod{16} \\ x &\equiv 2 \pmod{33}. \end{aligned}$$

▶ **Solution.** Since $33 - 2 \cdot 16 = 1$, then $x = 9 \cdot 33 - 2(2 \cdot 16) = 233$ is one solution to the simultaneous congruences, and all solutions are given by $x = 233 + 33 \cdot 16k = 233 + 528k$ for $k \in \mathbb{Z}$. ◀

5. [**16 Points**] In $\mathbb{Z}_{35}$ find the inverse of $\overline{13}$ and use it to solve the equation $\overline{13}x = \overline{9}$.

▶ **Solution.** Use the Euclidean Algorithm (or inspection) to write $1 = 3 \cdot 35 - 8 \cdot 13$. This equation shows that $\overline{13}^{-1} = \overline{-8} = \overline{27}$.

Solve the equation $\overline{13}x = \overline{9}$ by multiplication by $\overline{13}^{-1}$ to get

$$x = \overline{13}^{-1} \cdot \overline{9} = \overline{-8} \cdot \overline{9} = \overline{-72} = \overline{33}.$$

◀

6. [**16 Points**] Let $\sigma = (1\ 2\ 5\ 4)(2\ 4\ 6)(3\ 6\ 4) \in S_6$.

(a) Write σ in two-row notation, i.e., complete the second row of the following matrix:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \sigma(1) & \sigma(2) & \sigma(3) & \sigma(4) & \sigma(5) & \sigma(6) \end{pmatrix}.$$

▶ **Solution.**

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 5 & 3 & 4 & 6 \end{pmatrix}.$$

◀

(b) Write σ as a product of *disjoint* cycles.

► **Solution.** $\sigma = (1\ 2)(3\ 5\ 4)$. ◄