Instructions. Answer each of the questions on your own paper, and be sure to show your work, including giving reasons, so that partial credit can be adequately assessed. Put your name on each page of your paper.

1. [**20 Points**]
   (a) State Lagrange's Theorem. Be sure to include any hypotheses.
   (b) Let G be a group with $|G| < 300$. If G has a subgroup H of order 24 and a subgroup K of order 54, what is the order of G? What are the possibilities for the order of $H \cap K$?

2. [**20 Points**]
   (a) Let G be a group and let $a \in G$. Complete the definition of what it means for a to have finite order $o(a) = n$: *If n is a positive integer, then the element $a \in G$ has order n provided*

   (b) Find the order of each of the following group elements. Justify your answer either by a calculation or by reference to an appropriate theorem.
      i. The element 2 in the group $\mathbb{Z}_7^*$.
      ii. The element $\sigma = \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3 & 4 & 5 & 6\end{pmatrix}$ in the group S_6.
      iii. The element g^8 in the cyclic group $G = \langle g \rangle$ with $o(g) = 20$.

3. [**20 Points**] Let $G = \langle a \rangle$ be a cyclic group of order 9 and let $H = S_3$.
   (a) Complete the following table so as to make the function $\alpha : G \to H$ a homomorphism. (*Hint:* Use the fact that a homomorphism satisfies $\alpha(a^2) = \alpha(a)\alpha(a)$, $\alpha(a^3) = \alpha(a^2)\alpha(a)$, etc.)

g	1	a	a^2	a^3	a^4	a^5	a^6	a^7	a^8
$\alpha(g)$	ε	$(1\ 2\ 3)$							

   (b) Find the kernel K of α. Recall that $K = \{g \in G \mid \alpha(g) = 1\}$ where 1 is the identity element.
   (c) Find the image $\alpha(G)$ of α.
   (d) Is α an isomorphism? Is G isomorphic to H? Note that these are not the same question.

4. [**20 Points**] Let $G = D_4 = \{1, a, a^2, a^3, b, ba, ba^2, ba^3\}$, where $o(a) = 4$, $o(b) = 2$, and $ab = ba^{-1}$, and let $H = \{1, b\}$, $K = \{1, a^3\}$ be subsets of G.
   (a) Verify that H is a subgroup of G, but that K is not a subgroup.
   (b) How many right cosets of H in G are there?
   (c) List all of the distinct right cosets of H in G. (Distinct means that you should list each coset only once.)
   (d) Is H a normal subgroup of G? Prove that your answer is correct.

5. [**20 Points**] Let $G = \mathbb{Z}_{13}^*$ be the group of multiplicatively invertible congruences classes modulo 13.
   (a) What is the order of G?
   (b) Prove that G is cyclic by showing that $G = \langle 2 \rangle$.
   (c) Using your answer to part (b), list all of the subgroups of G.