Instructions. Answer each of the questions on your own paper, and be sure to show your work, including giving reasons, so that partial credit can be adequately assessed. Put your name on each page of your paper.

1. **[20 Points]**

   (a) State Lagrange's Theorem. Be sure to include any hypotheses.

   ▶ **Solution.** *Let G be a finite group and let H be a subgroup. Then $|H|$ divides $|G|$.* ◀

   (b) Let G be a group with $|G| < 300$. If G has a subgroup H of order 24 and a subgroup K of order 54, what is the order of G? What are the possibilities for the order of $H \cap K$?

   ▶ **Solution.** By Lagrange's theorem, $|G|$ must be a multiple of $|H| = 24$ and $|K| = 54$. Thus, $|G|$ must be a multiple of the least common multiple of 24 and 54. But, the least common multiple of $24 = 8 \cdot 3$ and $54 = 2 \cdot 27$ is $27 \cdot 8 = 216$. The only multiple of 216 less than 300 is 216 so $|G| = 216$. $|H \cap K|$ must divide both $|H| = 24$ and $|K| = 54$, so it must be a divisor of the greatest common divisor of 24 and 54, which is 6. Thus, $|H| \in \{1, 2, 3, 6\}$. ◀

2. **[20 Points]**

   (a) Let G be a group and let $a \in G$. Complete the definition of what it means for a to have finite order $o(a) = n$: *If n is a positive integer, then the element $a \in G$ has order n provided*

   $\boxed{a^n = 1 \text{ and } n \text{ is the smallest positive integer such that } a^n = 1.}$

   (b) Find the order of each of the following group elements. Justify your answer either by a calculation or by reference to an appropriate theorem.

   i. The element 2 in the group $\mathbb{Z}_7^*$.

   ▶ **Solution.** $2^1 = 2$, $2^2 = 4$, $2^3 = 8 = 1 \in \mathbb{Z}_7^*$. Thus, $o(2) = 3$. ◀

   ii. The element $\sigma = \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3 & 4 & 5 & 6\end{pmatrix}$ in the group S_6.

   ▶ **Solution.** $o(\sigma) = 4$ since σ is the disjoint product of a 2 cycle and a 4 cycle, and 4 is the least common multiple of 2 and 4. ◀

   iii. The element g^8 in the cyclic group $G = \langle g \rangle$ with $o(g) = 20$.

   ▶ **Solution.** If $(g^8)^n = 1$, then $g^{8n} = 1$ so $20|8n$ so $5|2n$ and since 2 and 5 are relatively prime, it follows that $5|n$. Since $(g^8)^5 = g^{40} = (g^{20})^2 = 1$ it follows that $o(g^8) = 5$. ◀

3. **[20 Points]** Let $G = \langle a \rangle$ be a cyclic group of order 9 and let $H = S_3$.

   (a) Complete the following table so as to make the function $\alpha : G \to H$ a homomorphism. (*Hint:* Use the fact that a homomorphism satisfies $\alpha(a^2) = \alpha(a)\alpha(a)$, $\alpha(a^3) = \alpha(a^2)\alpha(a)$, etc.)

g	1	a	a^2	a^3	a^4	a^5	a^6	a^7	a^8
$\alpha(g)$	ε	$(1\ 2\ 3)$	$(1\ 3\ 2)$	ε	$(1\ 2\ 3)$	$(1\ 3\ 2)$	ε	$(1\ 2\ 3)$	$(1\ 3\ 2)$

(b) Find the kernel K of α. Recall that $K = \{g \in G \mid \alpha(g) = 1\}$ where 1 is the identity element.

▶ **Solution.** $\mathrm{Ker}(\alpha) = \{1, a^3, a^6\}$. ◀

(c) Find the image $\alpha(G)$ of α. $\alpha(G) = \{\varepsilon, (1\ 2\ 3), (1\ 3\ 2)\}$.

(d) Is α an isomorphism? Is G isomorphic to H? Note that these are not the same question.

▶ **Solution.** α is not an isomorphism since it is not injective: $\alpha(1) = \alpha(a^3)$ but $a^3 \neq 1$. G is not isomorphic to H since $|G| = 9 \neq 6 = |H|$. Any isomorphism must be a bijection, so that means that the number of elements of G must be the same as the number of elements of H. ◀

4. **[20 Points]** Let $G = D_4 = \{1, a, a^2, a^3, b, ba, ba^2, ba^3\}$, where $o(a) = 4$, $o(b) = 2$, and $ab = ba^{-1}$, and let $H = \{1, b\}$, $K = \{1, a^3\}$ be subsets of G.

(a) Verify that H is a subgroup of G, but that K is not a subgroup.

▶ **Solution.** H is a subgroup since $b^2 = 1$ so H is closed under multiplication, and $b = b^{-1}$ so H is closed under inverses. Since $(a^3)^2 = a^6 = a^2 \notin K$, it follows that K is not closed under multiplication, and hence is not a subgroup. ◀

(b) How many right cosets of H in G are there?

▶ **Solution.** The number of right cosets is the index $[G : H] = |G| \,/\, |H| = 8/2 = 4$. ◀

(c) List all of the distinct right cosets of H in G. (Distinct means that you should list each coset only once.)

▶ **Solution.**

$$\begin{aligned} H &= \{1, b\} \\ Ha &= \{a, ba\} \\ Ha^2 &= \{a^2, ba^2\} \\ Ha^3 &= \{a^3, ba^3\}. \end{aligned}$$

◀

(d) Is H a normal subgroup of G? Prove that your answer is correct.

▶ **Solution.** H is not normal since $aH = \{a, ab = ba^3\} \neq Ha = \{a, ba\}$. ◀

5. **[20 Points]** Let $G = \mathbb{Z}_{13}^*$ be the group of multiplicatively invertible congruences classes modulo 13.

(a) What is the order of G?

▶ **Solution.** Since 13 is prime, $|\mathbb{Z}_{13}^*| = 13 - 1 = 12$. ◀

(b) Prove that G is cyclic by showing that $G = \langle 2 \rangle$.

► **Solution.** By Lagrange's theorem the order of 2 divides the order of G. Thus $o(2) \in \{1, 2, 3, 4, 6, 12\}$. But $2^1 \neq 1$, $2^2 = 4 \neq 1$, $2^3 = 8 \neq 1$, $2^4 = 16 = 3 \neq 1$, and $2^6 = (2^3)^2 = 8^2 = 64 = 12 \neq 1$. This means that the only possibility for $o(2)$ is 12, and thus, $G = \langle 2 \rangle$, so G is cyclic with generator 2. ◄

(c) Using your answer to part (b), list all of the subgroups of G.

► **Solution.** All of the subgroups of G are cyclic of the form $\langle 2^k \rangle$ where k is a divisor of 12. Thus, the distinct subgroups are $G = \langle 2 \rangle$, $\langle 2^2 = 4 \rangle$, $\langle 2^3 = 8 \rangle$, $\langle 2^4 = 3 \rangle$, $\langle 2^6 = 12 \rangle$, and $\langle 2^{12} = 1 \rangle$. ◄