Instructions. Answer each of the questions on your own paper and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.

1. [**20 Points**] Let $G = \mathbb{Z}_4 \times \mathbb{Z}_4$ and let $K = \langle(1, 2)\rangle$ be the cyclic subgroup of G generated by $(1, 2)$.
   (a) List all of the distinct cosets of K in G.
   (b) Write the multiplication table for the factor group G/K. (Remember the group operation of G is $+$.)
   (c) Is G/K a cyclic group? If so, find a generator, and show that your candidate is a generator. If not, show that it is not cyclic.

2. [**15 Points**] Complete **3** of the following **4** definitions.
   (a) A commutative ring R is called an *integral domain* if ...
   (b) A nonempty subset A of a ring R is an *ideal* of R if ...
   (c) If R is a commutative ring, and A is an ideal of R with $A \neq R$, then A is called a *prime ideal* of R if ...
   (d) If R is a ring, and A is an ideal of R with $A \neq R$, then A is called a *maximal ideal* of R if ...

3. [**20 Points**] Complete the following statements of theorems.
   (a) Let $\alpha : G \to H$ be a group homomorphism between the groups G and H. Then

$$G/\ker(\alpha) \cong \boxed{}.$$

   (b) Let $\theta : R \to S$ be a ring homomorphism between the rings R and S. Then

$$R/\ker(\theta) \cong \boxed{}.$$

   (c) Let A be an ideal of the commutative ring R with $A \neq R$. Then A is a *prime ideal* if and only if R/A is $\boxed{}$.
   (d) Let A be an ideal of the commutative ring R with $A \neq R$. Then A is a *maximal ideal* if and only if R/A is $\boxed{}$.

4. [**20 Points**] For the ring $\mathbb{Z}_{18}$ find
   (a) all units,
   (b) all nilpotent elements,
   (c) all ideals,
   (d) all prime ideals.

5. **[25 Points]** In the ring of Gaussian integers $\mathbb{Z}[i] = \{m + ni \mid m, n \in \mathbb{Z}\}$ let $A = \langle 3+i \rangle$ be the principal ideal generated by $3 + i$. That is,

$$A = (3 + i)\mathbb{Z}[i] = \{(3 + i)(m + ni) \mid m, n \in \mathbb{Z}\}.$$

(a) Show that $(m + ni) + A = (m - 3n) + A$ for all $m + ni \in \mathbb{Z}[i]$.

(b) Show that the ring homomorphism $\theta : \mathbb{Z} \to \mathbb{Z}[i]/A$ given by $\theta(k) = k + A$ is onto.

(c) Verify that $\mathrm{Ker}(\theta) = 10\mathbb{Z}$.

(d) Using (b) and (c) show that $\mathbb{Z}[i]/A \cong \mathbb{Z}_{10}$.

(e) Now show that A is not a prime ideal.