Instructions. Answer each of the questions on your own paper and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.

1. [**20 Points**] Let $G = \mathbb{Z}_4 \times \mathbb{Z}_4$ and let $K = \langle(1, 2)\rangle$ be the cyclic subgroup of G generated by $(1, 2)$.

   (a) List all of the distinct cosets of K in G.

   ▶ **Solution.** $K = \{(0, 0), (1, 2), (2, 0), (3, 2)\}$, so the cosets are

$$\begin{aligned}
K_1 = K + (0, 0) &= \{(0, 0), (1, 2), (2, 0), (3, 2)\} \\
K_2 = K + (1, 0) &= \{(1, 0), (2, 2), (3, 0), (0, 2)\} \\
K_3 = K + (0, 1) &= \{(0, 1), (1, 3), (2, 1), (3, 3)\} \\
K_4 = K + (1, 1) &= \{(1, 1), (2, 3), (3, 1), (0, 3)\}
\end{aligned}$$

◀

   (b) Write the multiplication table for the factor group G/K. (Remember the group operation of G is $+$.)

   ▶ **Solution.**

$+$	K_1	K_2	K_3	K_4
K_1	K_1	K_2	K_3	K_4
K_2	K_2	K_1	K_4	K_3
K_3	K_3	K_4	K_2	K_1
K_4	K_4	K_3	K_1	K_2

◀

   (c) Is G/K a cyclic group? If so, find a generator, and show that your candidate is a generator. If not, show that it is not cyclic.

   ▶ **Solution.** Since $\langle K_3\rangle = \{K_3, K_3 + K_3 = K_2, K_3 + K_2 = K_4, K_- + K_4 = K_1\} = G/K$, it follows that G/K is cyclic with generator $K_3 = K + (0, 1)$. ◀

2. [**15 Points**] Complete **3** of the following **4** definitions.

   (a) A commutative ring R is called an *integral domain* if ... $1 \neq 0$ and if $ab = 0$ implies that $a = 0$ or $b = 0$.

   (b) A nonempty subset A of a ring R is an *ideal* of R if ... for all $a, b \in A$, $a \pm b \in A$, and for all $a \in A$, $r \in R$, $ar \in A$ and $ra \in A$.

   (c) If R is a commutative ring, and A is an ideal of R with $A \neq R$, then A is called a *prime ideal* of R if ... for all $a, b \in R$, $ab \in A$ implies that $a \in A$ or $b \in B$.

   (d) If R is a ring, and A is an ideal of R with $A \neq R$, then A is called a *maximal ideal* of R if ... for any ideal B of R with $A \subseteq B \subseteq R$, either $B = A$ or $B = R$.

3. **[20 Points]** Complete the following statements of theorems.

   (a) Let $\alpha : G \to H$ be a group homomorphism between the groups G and H. Then

   $$G/\ker(\alpha) \cong \boxed{\alpha(G)}.$$

   (b) Let $\theta : R \to S$ be a ring homomorphism between the rings R and S. Then

   $$R/\ker(\theta) \cong \boxed{\theta(R)}.$$

   (c) Let A be an ideal of the commutative ring R with $A \neq R$. Then A is a *prime ideal* if and only if R/A is $\boxed{\text{an integral domain}}$.

   (d) Let A be an ideal of the commutative ring R with $A \neq R$. Then A is a *maximal ideal* if and only if R/A is $\boxed{\text{a field}}$.

4. **[20 Points]** For the ring $\mathbb{Z}_{18}$ find

   (a) all units,

   ▶ **Solution.** a is a unit if and only if $\gcd(a, 18) = 1$. Thus, the units are $\{1, 5, 7, 11, 13, 17\}$. ◀

   (b) all nilpotent elements,

   ▶ **Solution.** a is nilpotent if and only if $a^k = 0 \in \mathbb{Z}_{18}$ for some $k \geq 1$. But this is true if and only if 18 divides a^k in the integers. This means that 2 and 3 must divide a^k and hence, they must also divide a. A necessary condition for a to be nilpotent is that 6 divides a. Thus, the possible nilpotent elements are 0, 6, and 12. Since $6^2 = 36 = 0 \in \mathbb{Z}_{18}$, 6 is nilpotent. Also, $12^2 = 144 = 0 \in \mathbb{Z}_{18}$, so 12 is also nilpotent. Hence, the nilpotent elements are 0, 6, and 12. ◀

   (c) all ideals,

   ▶ **Solution.** The ideals of $\mathbb{Z}_{18}$ are all ideals of the form $m\mathbb{Z}_{18}$ where m is a divisor of 18. Thus, the different ideals are: $\{0\} = 0\mathbb{Z}_{18} = 18\mathbb{Z}_{18}$, $2\mathbb{Z}_{18}$, $3\mathbb{Z}_{18}$, $6\mathbb{Z}_{18}$, and $9\mathbb{Z}_{18}$. ◀

   (d) all prime ideals.

   ▶ **Solution.** The prime ideals are the ideals $m\mathbb{Z}_{18}$ where m is a prime divisor of 18. Thus, the prime ideals are $2\mathbb{Z}_{18}$ and $3\mathbb{Z}_{18}$. ◀

5. **[25 Points]** In the ring of Gaussian integers $\mathbb{Z}[i] = \{m + ni \mid m,\, n \in \mathbb{Z}\}$ let $A = \langle 3+i \rangle$ be the principal ideal generated by $3 + i$. That is,

$$A = (3 + i)\mathbb{Z}[i] = \{(3 + i)(m + ni) \mid m,\, n \in \mathbb{Z}\}\,.$$

(a) Show that $(m + ni) + A = (m - 3n) + A$ for all $m + ni \in \mathbb{Z}[i]$.

▶ **Solution.** $(m+ni)-(m-3n) = n(i+3)$ so $(m+ni)+A = (m-3n)+A$. ◀

(b) Show that the ring homomorphism $\theta : \mathbb{Z} \to \mathbb{Z}[i]/A$ given by $\theta(k) = k + A$ is onto.

▶ **Solution.** Let $m+ni+A \in \mathbb{Z}[i]/A$ be arbitrary. Then by part (a), $(m+ni)+A = (m - 3n) + A = \theta(k)$ where $k = m - 3n$. Thus, θ is onto. ◀

(c) Verify that $\mathrm{Ker}(\theta) = 10\mathbb{Z}$.

▶ **Solution.** Suppose $k \in \mathrm{Ker}(\theta)$. This means that $k+A = \theta(k) = 0+A$ so that $k \in A$. Therefore, $k = (3 + i)(a + bi) = (3a - b) + (a + 3b)i$ for some $a,\, b \in \mathbb{Z}$. Thus, $k = 3a-b$ and $0 = a+3b$. Solving the second equation for a gives $a = -3b$, and substitute in the first equation to give $k = 3a - b = 3(-3b) - b = -10b = 10(-b) \in 10\mathbb{Z}$. Therefore, $\mathrm{Ker}(\theta) \subseteq 10\mathbb{Z}$. Since $10 = (3 + i)(3 - 1)$ it follows that $10 \subseteq \mathrm{Ker}(\theta)$ and hence $10\mathbb{Z} \subseteq \mathrm{Ker}(\theta)$. ◀

(d) Using (b) and (c) show that $\mathbb{Z}[i]/A \cong \mathbb{Z}_{10}$.

▶ **Solution.** By part (c), $\mathbb{Z}/\operatorname{Ker}(\theta) = \mathbb{Z}/10\mathbb{Z} \cong \mathbb{Z}_{10}$. By part (b), $\theta(\mathbb{Z}) = \mathbb{Z}[i]/A$. Combining these with the ring isomorphism theorem gives $\mathbb{Z}[i]/A = \theta(\mathbb{Z}) \cong \mathbb{Z}/\operatorname{Ker}(\theta) \cong \mathbb{Z}_{10}$. ◀

(e) Now show that A is not a prime ideal.

▶ **Solution.** In $\mathbb{Z}_{10}$, $2 \cdot 5 = 10 = 0$ and $2 \neq 0$, $5 \neq 0$. Thus, $\mathbb{Z}_{10}$ is not an integral domain. But, $\mathbb{Z}[i]/A \cong \mathbb{Z}_{10}$ so $\mathbb{Z}[i]/A$ is not an integral domain, and thus, A is not a prime ideal. ◀