Instructions. Answer each of the questions on your own paper. Be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.

1. [**14 Points**] Circle True (T) or False (F). No reasons are needed for this problem.

(a)	T	F	A group of order 23 is isomorphic to $\mathbb{Z}_{23}$.
(b)	T	F	If K is a field then the direct product $K \times K$ is a field.
(c)	T	F	The group $\mathbb{Z}_4$ is a cyclic group.
(d)	T	F	A subgroup of a cyclic group is cyclic.
(e)	T	F	The product of 2 transpositions in S_n is an odd cycle.
(f)	T	F	If R is a ring and $ab = ac$, then $b = c$.
(g)	T	F	The ring $\mathbb{Z}_{17}$ is a field.

2. [**12 Points**] Define the following concepts:

   (a) H is a *normal* subgroup of a group G and K is the *factor group* G/H.

   (b) I is an *ideal* of a ring R.

   (c) $\theta : R \to S$ is a *ring homomorphism* from a ring R to a ring S.

3. [**14 Points**]

   (a) Compute the greatest common divisor d of the integers 1776 and 1492.

   (b) Write d as a linear combination $d = 1776 \cdot s + 1492 \cdot t$.

4. [**14 Points**] Solve the equation $5x = 23$ in the ring $\mathbb{Z}_{32}$.

5. [**15 Points**] Let $\sigma = (1\ 2)(5\ 8)(3\ 4\ 6)(5\ 2)(4\ 1)(3\ 7)(6\ 7)$ in S_8.

   (a) Write σ in two-rowed notation.

   (b) Write σ as a product of disjoint cycles.

   (c) Compute the order of σ in the group S_8.

   (d) Determine whether σ is even or odd.

   (e) Compute σ^{-1}. You may express your answer in whatever form you wish.

6. [**15 Points**] Let G be a group.

   (a) If H is a nonempty subset of G, state the conditions that need to be checked to verify that H is a *subgroup* of G.

   (b) If G is an abelian group, prove that

   $$H = \left\{x \in G \mid x^2 = 1\right\}$$

   is a subgroup of G.

   (c) Show that $H = \{x \in S_3 \mid x^2 = 1\}$ is *not* a subgroup of S_3. What does this say about the hypothesis in part (b) that G is abelian?

7. **[12 Points]** Use induction to prove that $(ab)^n = a^n b^n$ for all a, b in an abelian group G and $n \in \mathbb{N}$.

8. **[15 Points]** Find all the units, nilpotents, and idempotents in the ring $R = \mathbb{Z}_{24}$.

9. **[12 Points]**

   (a) Find a cyclic subgroup H of S_4 of order 4.

   (b) Find a non-cyclic subgroup K of S_4 of order 4.

10. **[12 Points]** Let $G = \langle a \rangle$ be a cyclic group of order 8 with generator a, with the group operation written multiplicatively, and let $H = \mathbb{Z}_2 \times \mathbb{Z}_4$, the cartesian product of $\mathbb{Z}_2$ and $\mathbb{Z}_4$. Remember that the group operation on the group $\mathbb{Z}_n$ is addition of congruence classes.

    (a) Complete the following table so as to make the function $\theta : G \to H$ a homomorphism. (*Hint:* Use the fact that a homomorphism satisfies $\theta(a^2) = \theta(a)\theta(a)$, $\theta(a^3) = \theta(a^2)\theta(a)$, etc. and remember that product means group operation.)

g	1	a	a^2	a^3	a^4	a^5	a^6	a^7
$\theta(g)$	$(\overline{0}, \overline{0})$	$(\overline{1}, \overline{3})$						

    (b) Find the kernel of θ. (Recall that $\mathrm{Ker}(\theta) = \{x \in G : \theta(x) = e'\}$, where e' is the identity of the group H.)

    (c) Find the image of θ. (Recall that $\mathrm{Im}(\theta) = \theta(G) = \{y \in H : y = \theta(x) \text{ for some } x \in G\}$.)

11. **[15 Points]** Let $G = \mathbb{Z}_{32}^* = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31\}$ and let $H = \langle 9 \rangle = \{1, 9, 17, 25\}$.

    (a) List all of the elements of the factor group G/H.

    (b) Write the multiplication table for G/H.

    (c) Every group of order 4 is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. You may assume this fact. Which of these two groups is G/H isomorphic to, and why?