Instructions. Answer each of the questions on your own paper, and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.

1. [**15 Points**] Let $m = 715$ and $n = 546$.
   (a) Calculate the greatest common divisor $d = \gcd(m, n)$.
   (b) Write d in the form $sm + tn$ for some integers s and t.
   (c) Calculate the least common multiple $\operatorname{lcm}(m, n)$.

2. [**20 Points**] Use induction to prove that the equation
$$\sum_{k=1}^{n}(4k-1) = 2n^2 + n$$
is valid for all integers $n \geq 1$. Note that expanding the summation, the equation is
$$3 + 7 + 11 + \cdots + (4n-1) = 2n^2 + n.$$

3. [**15 Points**] Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} A$ satisfy $\beta\alpha = 1_A$.
   (a) Show that β is onto.
   (b) Give an example for which β does not have an inverse.

4. [**20 Points**] This exercise makes use of the following equation:
$$1 = 6 \cdot 111 - 19 \cdot 35.$$
Using this equation (i.e., *do not* use the Euclidean algorithm to recreate it), answer the following questions.
   (a) Compute the multiplicative inverse of $\overline{35}$ in $\mathbb{Z}_{111}$. Express your answer in the standard form $\overline{a}$ where $0 \leq a < 111$.
   (b) Solve the equation $\overline{35}x = \overline{9}$ in $\mathbb{Z}_{111}$. Express your answer in the standard form $x = \overline{b}$ where $0 \leq b < 111$.
   (c) Find the smallest positive solution of the system of simultaneous linear congruences:
$$\begin{aligned} x &\equiv 6 \pmod{111} \\ x &\equiv 5 \pmod{19}. \end{aligned}$$

5. [**15 Points**] Let $A = \mathbb{R} \times \mathbb{R}$ and define a relation $\equiv$ on A by $(a, b) \equiv (a_1\, b_1)$ if $a^2 + b^2 = a_1^2 + b_1^2$.
   (a) Verify that $\equiv$ is an equivalence relation on A.
   (b) Find the equivalence class $[(1, 0)]$.
   (c) More generally, for each $(a, b) \in \mathbb{R} \times \mathbb{R}$, give a simple geometric description of the equivalence class $[(a, b)]$.

6. [**20 Points**] Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 4 & 3 & 2 & 7 & 6 & 1 & 5 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 2 & 7 & 8 & 1 & 6 & 4 \end{pmatrix}$.
   (a) Write σ as a product of disjoint cycles.
   (b) Write σ as a product of transpositions. Recall that a transposition is a 2-cycle.
   (c) Compute $\sigma\tau$ and $\tau\sigma$.