Instructions. Answer each of the questions on your own paper, and be sure to show your work, including giving reasons, so that partial credit can be adequately assessed. Put your name on each page of your paper.

1. **[30 Points]** These are short answer questions. For these questions, you do not need to show your work.

   (a) List the generators of $\mathbb{Z}_{12}$. .... $\boxed{1, 5, 7, 11}$

   (b) Write a multiplication table for the group $\mathbb{Z}_{12}^*$. ....

$\cdot$	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

   (c) What is the order of 3 in $\mathbb{Z}_{11}^*$? .... $\boxed{5}$

   (d) $|S_4| = ?$ .... $\boxed{4! = 24}$

   (e) A group has 45 elements. What are the possible orders of its subgroups? $\boxed{1, 3, 5, 9, 15, 45}$

   (f) Suppose that a is a generator of a cyclic group G of order 45. What is the order of the subgroup $\langle a^9 \rangle$. .... $\boxed{5}$

   (g) Suppose that a is a generator of a cyclic group G. Give a generator for the subgroup $\langle a^{12} \rangle \cap \langle a^{10} \rangle$. .... $\boxed{\langle a^{60} \rangle}$

   (h) Give an example of a nontrivial (meaning not equal to $\langle 1 \rangle$) abelian subgroup of a non-abelian group. .... $\boxed{\{\epsilon, (1\ 2)\} \subset S_3}$

   (i) Write $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 1 & 6 & 7 & 2 & 4 & 3 \end{pmatrix} \in S_7$ as a product of disjoint cycles. $\boxed{(1\ 5\ 2)(3\ 6\ 4\ 7)}$

   (j) Find the order of σ from part (i). .... $\boxed{\text{lcm}(3,\ 4) = 12}$

2. **[16 Points]** Let $G = \left\{ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},\ B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$, and let $H = \left\{ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},\ D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},\ E = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$ be two subsets of 2×2 matrices.

   (a) The following tables are multiplication tables for G and H respectively:

   G

$\cdot$	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	A	I
C	C	B	I	A

   H

$\cdot$	I	A	D	E
I	I	A	D	E
A	A	I	E	D
D	D	E	I	A
E	E	D	A	I

   Explain briefly why the above multiplication tables show that G and H are subgroups of the group $\mathrm{GL}_2(\mathbb{R})$ of invertible 2×2 matrices with group operation matrix multiplication.

   ▶ **Solution.** The multiplication tables show that both G and H are finite subsets of $\mathrm{GL}_2(\mathbb{R})$ that are closed under multiplication, and hence are subgroups. ◀

(b) Verify that $K = \{I,\, B\}$ is *not* a subgroup of G.

▶ **Solution.** $B^2 = A \notin K$ so K is not closed under multiplication, and hence is not a subgroup. ◀

(c) The set K is not a subgroup of G, but there is a subgroup of G consisting of 2 elements. Which two elements of G form a subgroup?

▶ **Solution.** $\{I,\, A\}$ is closed under multiplication since $A^2 = I$ and hence is a subgroup of G. ◀

(d) Determine whether G and H are isomorphic.

▶ **Solution.** An isomorphism must preserve the orders of elements. Since G has an element of order 4, namely B, but all elements of H have order 2. Thus, G and H are not isomorphic. ◀

3. [**14 Points**]

(a) State Lagrange's Theorem. Be sure to include any hypotheses.

▶ **Solution.** If G is a finite group and H is a subgroup, then $|H| \mid |G|$. ◀

(b) What is the relationship between the order of an element $g \in G$ and the order of the group G?

▶ **Solution.** The order of an element $g \in G$ divides the order of the group G. ◀

(c) If $g \in G$ is an element such that $g \neq 1$ and $g^{18} = g^{33} = 1$, then $o(g) =$?

▶ **Solution.** Since $g^{18} = 1$, $o(g)|18$ and since $g^{33} = 1$, $o(g)|33$. Hence, $o(g)$ is a common divisor of 18 and 33, and thus is a divisor of $\gcd(18,\, 33) = 3$. Thus, $o(g) = 1$ or $o(g) = 3$. Since $g \neq 1$, we must have $o(g) = 3$. ◀

4. [**10 Points**] List the right cosets of the subgroup $H = \left\{\varepsilon,\, \begin{pmatrix}1 & 3\end{pmatrix}\right\}$ of S_3.

▶ **Solution.** $H\varepsilon = \{\varepsilon,\, (1\ 3)\}$, $H(1\ 2) = \{(1\ 2),\, (1\ 2\ 3)\}$, $H(2\ 3) = \{(2\ 3), (1\ 3\ 2)\}$. ◀

5. [**14 Points**] Let $G = \mathbb{Z}_8$ and let $H = S_4$.

(a) Complete the following table so as to make the function $\alpha : G \to H$ a homomorphism. Remember that addition is the operation of $G = \mathbb{Z}_8$, while multiplication is the operation of $H = S_4$, so that the homomorphism property of α is $\alpha(r + s) = \alpha(r)\alpha(s)$.

g	0	1	2	3	4	5	6	7
$\alpha(g)$	ε	$(1\ 4\ 3\ 2)$	$(1\ 3)(4\ 2)$	$(1\ 2\ 3\ 4)$	ε	$(1\ 4\ 3\ 2)$	$(1\ 3)(4\ 2)$	$(1\ 2\ 3\ 4)$

(b) Find the kernel K of α. Recall that $K = \{g \in G \mid \alpha(g) = 1\}$ where 1 is the identity element.

▶ **Solution.** $K = \{0,\, 4\}$ ◀

(c) Find the image $\alpha(G)$ of α.

▶ **Solution.** $\alpha(G) = \{\varepsilon,\ (1\ 4\ 3\ 2),\ (1\ 3)(4\ 2),\ (1\ 2\ 3\ 4)\}$ ◀

6. **[16 Points]** Suppose that a group G has the property that $g^2 = 1$ for every $g \in G$. Prove that G is abelian.

▶ **Solution.** Since $g^2 = 1$ for every $g \in G$, it follows that $g = g^{-1}$ for every $g \in G$. Thus, if g and h are arbitrary elements of G, then

$$gh = (gh)^{-1} = h^{-1}g^{-1} = hg,$$

and G is abelian. ◀