Instructions. Answer each of the questions on your own paper and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.

1. [**20 Points**] Let $G = \mathbb{Z}_4 \times \mathbb{Z}_8$ and let $K = \langle(1, 4)\rangle$ be the cyclic subgroup of G generated by $(1, 4)$.
   (a) List all of the elements of K.
   (b) What is the order of the factor group G/K?
   (c) What is the order of the element $g = (1, 1) + K$ in the factor group G/K. (Remember the group operation of G is $+$.)
   (d) Is G/K a cyclic group? Justify your answer.

2. [**20 Points**]
   (a) What properties must a non-empty subset S of a ring R satisfy in order to be a subring?
   (b) Define what it means for two rings R and R' to be isomorphic?
   (c) Prove that
   $$R_1 = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| \, a \in \mathbb{Q}, \, b \in \mathbb{Z} \right\}$$
   is a subring of $M_2(\mathbb{Q})$, the ring of 2×2 matrices with entries in the rational numbers $\mathbb{Q}$.
   (d) Prove that $R_1 \cong \mathbb{Z} \times \mathbb{Q}$.

3. [**14 Points**]
   (a) What properties must a subset I of a ring R satisfy in order to be an ideal?
   (b) Prove that $I_1 = \left\{ \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} \middle| \, a, \, c \in \mathbb{Z} \right\}$ is an ideal of $R_1 = \left\{ \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \middle| \, A, \, C, \, D \in \mathbb{Z} \right\}$.
   (c) Is I_1 an ideal of $M_2(\mathbb{Z})$? Why or why not.

4. [**16 Points**] All of the following are commutative rings.
   $$R_1 = \mathbb{Z}_5, \quad R_2 = \mathbb{Z}_6 \times \mathbb{Z}_4, \quad R_3 = \{a + bi \mid a, \, b \in \mathbb{Z}\}, \quad R_4 = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \middle| \, a \in \mathbb{Q} \right\}.$$
   (a) Give the identity element 1_{R_j} for $j = 1, 2, 3, 4$.
   (b) What is the characteristic $\mathrm{char}(R_j)$ for each of the rings R_j, $j = 1, 2, 3, 4$.
   (c) Which of the rings, if any, are integral domains?
   (d) Which of the rings, if any, are fields?

5. [**20 Points**] Suppose that R is a commutative ring.

   (a) Define what it means for an ideal I to be a *prime* ideal.

   (b) Define what it means for an ideal I to be a *maximal* ideal.

   (c) Is the ideal $2\mathbb{Z} \times 5\mathbb{Z}$ a prime ideal in $\mathbb{Z} \times \mathbb{Z}$? If yes, give a proof. If not, show why not.

   (d) Is the ideal $\{0\} \times \mathbb{Z}$ a prime ideal in $\mathbb{Z} \times \mathbb{Z}$? If yes, give a proof. If not, show why not.

   (e) Is the ideal $\{0\} \times \mathbb{Z}$ a maximal ideal in $\mathbb{Z} \times \mathbb{Z}$? If yes, give a proof. If not, show why not.

6. [**10 Points**] Indicate if the statement is True (**T**) for False (**F**). For this problem, reasons are not required.

   (a) $\mathbb{Z}_{17}$ is a field.

   (b) $\mathbb{N}$ is a subring of $\mathbb{Z}$. (Recall $\mathbb{N} = \{n \in \mathbb{Z} \mid n \geq 0\}$.)

   (c) The order of Kg in G/K is the smallest n such that $g^n = 1_G$.

   (d) In a field F, $ac = bc$ implies that $a = b$.

   (e) If $\phi : G \to G'$ is an onto group homomorphism with kernel K then $G/K \cong G'$.