Instructions. Answer each of the questions on your own paper and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.

1. **[20 Points]** Let $G = \mathbb{Z}_4 \times \mathbb{Z}_8$ and let $K = \langle(1, 4)\rangle$ be the cyclic subgroup of G generated by $(1, 4)$.

   (a) List all of the elements of K. (Remember the group operation of G is $+$.)

   ▶ **Solution.** $K = \{(0, 0), (1, 4), (2, 0), (3, 4)\}$ ◀

   (b) What is the order of the factor group G/K?

   ▶ **Solution.** $|G| = 4 \times 8 = 32$ and $|K| = 4$, so by Lagrange's theorem $|G/K| = |G| / |K| = 32/4 = 8$. ◀

   (c) What is the order of the element $g = (1, 1) + K$ in the factor group G/K.

   ▶ **Solution.** Since $|G/K| = 8$, $o(g)|8$ so $o(g) = 1, 2, 4$, or 8. Note that $4g = 4((1, 1) + K) = (4, , 4) + K = (0, 4) + K \neq (0, 0) + K$ since $(0, 4) \notin K$. Thus, $o(g) \neq 1, 2$, or 4, so that $o(g) = 8$. ◀

   (d) Is G/K a cyclic group? Justify your answer.

   ▶ **Solution.** Since $|G/K| = 8$ and $|\langle g\rangle| = o(g) = 8$ it follows that G/K is cyclic with generator $g = (1, 1) + K$. ◀

2. **[20 Points]**

   (a) What properties must a non-empty subset S of a ring R satisfy in order to be a subring?

   ▶ **Solution.** S must contain 0 and 1 and if $r, s \in S$, then S must also contain $r + s$, $-r$, and rs. ◀

   (b) Define what it means for two rings R and R' to be isomorphic?

   ▶ **Solution.** R is isomorphic to R' provided there is a one-to-one and onto function $\theta : R \to R'$ such that for all r, s in R, $\theta(r + s) = \theta(r) + \theta(s)$ and $\theta(rs) = \theta(r)\theta(s)$. ◀

   (c) Prove that

   $$R_1 = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| \, a \in \mathbb{Q},\ b \in \mathbb{Z} \right\}$$

   is a subring of $M_2(\mathbb{Q})$, the ring of 2×2 matrices with entries in the rational numbers $\mathbb{Q}$.

▶ **Solution.** Use the subring test from part (a). $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in R_1$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in R_1$. Suppose that $r = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and $s = \begin{bmatrix} a' & 0 \\ 0 & b' \end{bmatrix} \in R_1$. Thus a, $a' \in \mathbb{Q}$ and b, $b' \in \mathbb{Z}$, so that $a + a'$, $-a$, $aa' \in \mathbb{Q}$ and $b + b'$, $-b$, $bb' \in \mathbb{Z}$. Then

$$\begin{aligned} r + s &= \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} a' & 0 \\ 0 & b' \end{bmatrix} = \begin{bmatrix} a + a' & 0 \\ 0 & b + b' \end{bmatrix} \in R_1 \\ -r &= -\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix} \in R_1 \\ rs &= \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} a' & 0 \\ 0 & b' \end{bmatrix} = \begin{bmatrix} aa' & 0 \\ 0 & bb' \end{bmatrix} \in R_1 \end{aligned}$$

◀

(d) Prove that $R_1 \cong \mathbb{Z} \times \mathbb{Q}$.

▶ **Solution.** Define $\theta : R_1 \to \mathbb{Z} \times \mathbb{Q}$ by $\theta\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = (b,\, a)$. It is clear that θ is one-to-one and onto. Then, $\theta\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = (1,\, 1)$, so θ takes the identity of R_1 to the identity of $\mathbb{Z} \times \mathbb{Q}$. Moreover,

$$\begin{aligned} \theta\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} a' & 0 \\ 0 & b' \end{bmatrix}\right) &= \theta\left(\begin{bmatrix} a + a' & 0 \\ 0 & b + b' \end{bmatrix}\right) = (b + b',\, a + a') \\ &= (b,\, a) + (b',\, a') = \theta\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) + \theta\left(\begin{bmatrix} a' & 0 \\ 0 & b' \end{bmatrix}\right) \end{aligned}$$

and

$$\begin{aligned} \theta\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} a' & 0 \\ 0 & b' \end{bmatrix}\right) &= \theta\left(\begin{bmatrix} aa' & 0 \\ 0 & bb' \end{bmatrix}\right) = (bb',\, aa') \\ &= (b,\, a)(b',\, a') = \theta\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) \theta\left(\begin{bmatrix} a' & 0 \\ 0 & b' \end{bmatrix}\right). \end{aligned}$$

Therefore, θ is a ring isomorphism and $R_1 \cong \mathbb{Z} \times \mathbb{Q}$. ◀

3. [**14 Points**]

(a) What properties must a subset I of a ring R satisfy in order to be an ideal?

▶ **Solution.** I is an ideal of R provided that whenever a, $b \in I$ and $r \in R$, then $a \pm b \in I$, $ar \in I$ and $ra \in I$. ◀

(b) Prove that $I_1 = \left\{ \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} \,\middle|\, a,\, c \in \mathbb{Z} \right\}$ is an ideal of $R_1 = \left\{ \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \,\middle|\, A,\, C,\, D \in \mathbb{Z} \right\}$.

▶ **Solution.** Let $\alpha = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}$ and $\beta = \begin{bmatrix} a' & 0 \\ c' & 0 \end{bmatrix}$ be in I_1 and let $r = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \in R_1$. Then

$$\alpha \pm \beta = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} \pm \begin{bmatrix} a' & 0 \\ c' & 0 \end{bmatrix} = \begin{bmatrix} a \pm a' & 0 \\ c \pm b' & 0 \end{bmatrix} \in I_1,$$

$$r\alpha = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} aA & 0 \\ Ca + Dc & 0 \end{bmatrix} \in I_1,$$

and

$$\alpha r = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} = \begin{bmatrix} aA & 0 \\ cA & 0 \end{bmatrix} \in I_1.$$

Therefore, I_1 is an ideal of R_1. ◀

(c) Is I_1 an ideal of $M_2(\mathbb{Z})$? Why or why not.

▶ **Solution.** I_1 is not an ideal of $M_2(\mathbb{Z})$ since $\alpha = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in I_1$, but for $r = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in M_2(\mathbb{Z})$,

$$\alpha r = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \notin I_1.$$

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4. **[16 Points]** All of the following are commutative rings.

$$R_1 = \mathbb{Z}_5, \quad R_2 = \mathbb{Z}_6 \times \mathbb{Z}_4, \quad R_3 = \{a + bi \mid a,\, b \in \mathbb{Z}\}, \quad R_4 = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \middle|\, a \in \mathbb{Q} \right\}.$$

(a) Give the identity element 1_{R_j} for $j = 1,\, 2,\, 3,\, 4$.

▶ **Solution.** $1_{R_1} = 1$, $1_{R_2} = (1,\, 1)$, $1_{R_3} = 1$, $1_{R_4} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ◀

(b) What is the characteristic $\operatorname{char}(R_j)$ for each of the rings R_j, $j = 1,\, 2,\, 3,\, 4$.

▶ **Solution.** $\operatorname{char}(R_1) = 5$, $\operatorname{char}(R_2) = \operatorname{lcm}(6,\, 4) = 12$, $\operatorname{char}(R_3) = 0$, $\operatorname{char}(R_4) = 0$. ◀

(c) Which of the rings, if any, are integral domains?

▶ **Solution.** $R_1 = \mathbb{Z}_5$ is an integral domain since 5 is a prime. In R_2, $(1,\, 0)(0,\, 1) = (0,\, 0)$ with $(1,\, 0) \neq (0,\, 0)$ and $(0,\, 1) \neq (0,\, 0)$ so R_2 is not an integral domain. R_3 is a subring of the field $\mathbb{C}$ so it is an integral domain. $R_4 = \mathbb{Q} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ so $R_4 \cong \mathbb{Q}$ and hence R_4 is an integral domain. ◀

(d) Which of the rings, if any, are fields?

▶ **Solution.** R_1 and R_4 are fields. The others are not. ◀

5. [**20 Points**] Suppose that R is a commutative ring.

   (a) Define what it means for an ideal I to be a *prime* ideal.

   ▶ **Solution.** An ideal I is prime if $I \neq R$ and if $ab \in I$ implies that $a \in I$ or $b \in I$. ◀

   (b) Define what it means for an ideal I to be a *maximal* ideal.

   ▶ **Solution.** I is maximal provided $I \neq R$ and if J is an ideal of R with $I \subseteq J \subseteq R$ then $J = I$ or $J = R$. ◀

   (c) Is the ideal $2\mathbb{Z} \times 5\mathbb{Z}$ a prime ideal in $\mathbb{Z} \times \mathbb{Z}$? If yes, give a proof. If not, show why not.

   ▶ **Solution.** This is not a prime ideal since $(2,\, 1)(1,\, 5) = (2,\, 5) \in 2\mathbb{Z} \times 5\mathbb{Z}$ but neither $(2,\, 1)$ or $(1,\, 5)$ are in $2\mathbb{Z} \times 5\mathbb{Z}$. ◀

   (d) Is the ideal $\{0\} \times \mathbb{Z}$ a prime ideal in $\mathbb{Z} \times \mathbb{Z}$? If yes, give a proof. If not, show why not.

   ▶ **Solution.** This is a prime ideal since if $(a,\, b)(c,\, d) = (ac,\, bd) \in \{0\} \times \mathbb{Z}$, then $ac = 0$ and since $\mathbb{Z}$ is an integral domain this means that $a = 0$ or $c = 0$. Thus, either $(a,\, b)$ or $(c,\, d) \in \{0\} \times \mathbb{Z}$. ◀

   (e) Is the ideal $\{0\} \times \mathbb{Z}$ a maximal ideal in $\mathbb{Z} \times \mathbb{Z}$? If yes, give a proof. If not, show why not.

   ▶ **Solution.** This is not a maximal ideal since

   $$\{0\} \times \mathbb{Z} \subsetneqq 2\mathbb{Z} \times \mathbb{Z} \subsetneqq \mathbb{Z} \times \mathbb{Z}.$$

   ◀

6. [**10 Points**] Indicate if the statement is True (**T**) for False (**F**). For this problem, reasons are not required.

   (a) $\mathbb{Z}_{17}$ is a field. **T** since 17 is prime.

   (b) $\mathbb{N}$ is a subring of $\mathbb{Z}$. (Recall $\mathbb{N} = \{n \in \mathbb{Z} \mid n \geq 0\}$.) **F** since it is not closed under multiplication by -1.

   (c) The order of Kg in G/K is the smallest n such that $g^n = 1_G$. **F** The order of Kg is the smallest n such that $g^n \in K$.

   (d) In a field F, $ac = bc$ implies that $a = b$. **F** Not true if $c = 0$.

   (e) If $\phi : G \to G'$ is an onto group homomorphism with kernel K then $G/K \cong G'$. **T** This is the fundamental isomorphism theorem.