Instructions. Answer each of the questions on your own paper. True-False answers can be circled on this page. Be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.

1. [**12 Points**]

   (a) Assume that G is a set that is closed under a binary operation $*$. What properties are needed to make G a group with the binary operation $*$?

   (b) The set $S = \{a \in \mathbb{Q} \mid a \neq 0\}$ is closed under the commutative binary operation $*$ defined by
   $$a * b = \frac{ab}{3}.$$
   Prove that the set S with the binary operation is an abelian group.

2. [**14 Points**] Circle True (T) or False (F).

   T F (a) If G is a group with $|G| = 13$, then G and $\{1\}$ are the only subgroups.
   T F (b) If G is a group with $|G| = 12$, then the order of every element of G is 12.
   T F (c) If H is a subgroup of a group G with $|G| = 15$ and $|H| = 3$, then every coset of H has 5 elements.
   T F (d) $o(\sigma) \leq 5$ for every σ in the permutation group S_5.
   T F (e) The ring $\mathbb{Z}_3[x]/\langle x^2\rangle$ has nine elements, namely $a + bx + \langle x^2\rangle$, a, $b \in \mathbb{Z}_3$.
   T F (f) The ring $\mathbb{Q}[x]/\langle x^2 - 2\rangle$ is an integral domain.
   T F (g) The ring $\mathbb{R}[x]/\langle x^2 - 2\rangle$ is a field.

3. [**15 Points**]

   (a) What properties must a subset S of a ring R satisfy in order to be a subring?

   (b) Prove that $T = \{a + bi \mid a, b \in \mathbb{Q}\}$ is a subring of the ring $\mathbb{C}$ of complex numbers. (Recall $\mathbb{Q}$ denotes the rational numbers and $i = \sqrt{-1}$.)

   (c) Is T a field? Why or why not?

4. [**12 Points**]

   (a) Suppose that G is a finite group and H is a subgroup of G. Then Lagrange's theorem states that:

   (b) Assume that G is a finite group with subgroups H of order 12 and K of order 30. If the order of G is less than 200, what are the possible values for the order of G.

5. [**15 Points**] Suppose that R is a commutative ring.

   (a) What properties must a subset I of R satisfy in order to be an ideal?

   (b) Define what it means for an ideal to be *prime*.

   (c) Define what it means for an ideal to be *maximal*.

   (d) Show that $I_1 = 2\mathbb{Z} \times 3\mathbb{Z}$ is an ideal of the ring $\mathbb{Z} \times \mathbb{Z}$.

   (e) Show that $I_1 = 2\mathbb{Z} \times 3\mathbb{Z}$ is not a prime ideal of the ring $\mathbb{Z} \times \mathbb{Z}$.

6. **[14 Points]** For the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 8 & 7 & 2 & 4 & 9 & 1 & 5 & 6 \end{pmatrix}$:

   (a) Write σ as a product of disjoint cycles.

   (b) Write σ as a product of transpositions.

   (c) Is σ even, odd, neither or both?

   (d) What is the order of σ?

7. **[12 Points]**

   (a) If $G = \langle a \rangle$ is a cyclic group of order 100 then what is the order of a^{24}?

   (b) What is the order of the group $\mathbb{Z}_{40} \times \mathbb{Z}_{60}$?

   (c) What is the order of (4, 4) in $\mathbb{Z}_{40} \times \mathbb{Z}_{60}$?

8. **[12 Points]** Suppose that F is a field.

   (a) Suppose that $f(x)$, $g(x)$ are nonzero polynomials in $F[x]$. What does the division algorithm in $F[x]$ say?

   (b) Find the quotient $q(x)$ and remainder $r(x)$ when $g(x) = 3x^3 + x^2 + 2x + 3$ is divided by $f(x) = 2x^2 + 3$ in $\mathbb{Z}_5[x]$.

9. **[12 Points]**

   (a) Compute the greatest common divisor d of the integers 803 and 154.

   (b) Write d as a linear combination $d = 803 \cdot s + 154 \cdot t$.

   (c) Compute the least common multiple m of the integers 803 and 154.

10. **[10 Points]** Solve the equation $5x = 12$ in the ring $\mathbb{Z}_{44}$.

11. **[12 Points]**

    (a) List all of the subgroups of the cyclic group $\mathbb{Z}_{45}$ and give the order of each subgroup.

    (b) Draw the subgroup diagram for $\mathbb{Z}_{45}$.

12. **[16 Points]** Let $G = \mathbb{Z}_{21}^* = \{1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20\}$ and let $H = \langle 4 \rangle = \{1, 4, 16\}$.

    (a) Explain why H is a normal subgroup of G.

    (b) List all of the cosets of H in G. Label each coset as Hg where g is as small as possible. For example, the coset $\{1, 4, 16\}$ (which $= H1 = H4 = H16$) would be labeled $H1$. This is just for convenient labeling. How many cosets are there?

    (c) Write the multiplication table for G/H. List the elements of G/H as in part (b).

    (d) Every group of order 4 is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. You may assume this fact. Which of these two groups is G/H isomorphic to, and why?