Instructions. Answer each of the questions on your own paper, and be sure to show your work, including giving reasons, so that partial credit can be adequately assessed. Put your name on each page of your paper. There is a total possible of 75 points.

1. [**20 Points**] Let $\sigma \in S_9$ be the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 4 & 7 & 6 & 9 & 8 & 3 & 2 & 1 \end{pmatrix}.$$

(a) Express σ as a product of disjoint cycles. Is σ even or odd?

▶ **Solution.** $\sigma = \begin{pmatrix} 1 & 5 & 9 \end{pmatrix} \begin{pmatrix} 2 & 4 & 6 & 8 \end{pmatrix} \begin{pmatrix} 3 & 7 \end{pmatrix}$. An r-cycle is even if r is odd and odd if r is even. Thus, σ is the product of an even and two odd cycles, and hence is even, since an odd $\times$ odd permutation is even. ◀

(b) Find the order of σ.

▶ **Solution.** $|\sigma| = \operatorname{lcm}(3,\, 4,\, 2) = 12.$ ◀

(c) Express σ^{-1} as a product of disjoint cycles.

▶ **Solution.** $\sigma^{-1} = \begin{pmatrix} 3 & 7 \end{pmatrix} \begin{pmatrix} 2 & 8 & 6 & 4 \end{pmatrix} \begin{pmatrix} 1 & 9 & 5 \end{pmatrix}$ ◀

(d) Let $\tau = \begin{pmatrix} 1 & 2 \end{pmatrix}$. Compute $\tau\sigma$. Give your answer as a product of disjoint cycles.

▶ **Solution.**

$$\begin{aligned} \tau\sigma &= \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 5 & 9 \end{pmatrix} \begin{pmatrix} 2 & 4 & 6 & 8 \end{pmatrix} \begin{pmatrix} 3 & 7 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 5 & 9 & 2 & 4 & 6 & 8 \end{pmatrix} \begin{pmatrix} 3 & 7 \end{pmatrix} \end{aligned}$$

◀

2. [**15 Points**] Let G be a finite group with $|G| = 10$.

(a) What are the possible orders of elements of G?

▶ **Solution.** If $g \in G$ then $|g| \mid |G|$ so $|g| \mid 10 \implies |g| \in \{1,\, 2,\, 5,\, 10\}$. ◀

(b) If H is a subgroup of G and $H \neq G$, prove that H is cyclic.

▶ **Solution.** Since H is a subgroup of G, Lagrange's theorem gives that $|H| \mid |absG$, and if $H \neq G$, then $|H|$ must be 1, 2, or 5. But any group of prime order is cyclic (Corollary 6.12). Thus, if $|H| = 2$ or 5, then H is cyclic, and if $|H| = 1$, then $H\langle e \rangle$ so H is also cyclic in this case. ◀

(c) Give an example of a nonabelian group of order 10. (*Hint.* What is the order of the dihedral group D_n?)

▶ **Solution.** Since the dihedral group D_n has order $2n$, it follows that $|D_n| = 10$. This is a nonabelian group since the rotation r by $2\pi/5$ and the reflection s through the line joining a vertex and the midpoint of the opposite side satisfy $sr = r^{-1}s$ and $r \neq r^{-1}$. ◀

3. **[15 Points]** Let $G = U(15)$ be the group of integers modulo 15 that have a multiplicative inverse. Thus, $G = U(15) = \{1, 2, 4, 7, 8, 11, 13, 14\}$. Let $H = \langle 4 \rangle$ be the cyclic subgroup of G generated by 4.

   (a) Find $|H|$ and $[G : H]$, the number of cosets of H in G.

   ▶ **Solution.** $H = \{1, 4\}$ since $4^2 \equiv 1 \pmod{15}$. Thus, $|H| = 2$ and $[G : H] = |G| / |H| = 8/2 = 4$. ◀

   (b) Give a list of all of the *distinct* cosets of H in G.

   ▶ **Solution.** The distinct cosets are $1H = H = \{1, 4\}$, $2H = \{2, 8\}$, $7H = \{7, 13\}$, $11H = \{11, 14\}$. ◀

   (c) Is the factor group G/H a cyclic group? If so, find a generator. If not, show that it is not cyclic.

   ▶ **Solution.** The Cayley multiplication table for G/H is

$\cdot$	$1H$	$2H$	$7H$	$11H$
$1H$	$1H$	$2H$	$7H$	$11H$
$2H$	$2H$	$1H$	$11H$	$7H$
$7H$	$7H$	$11H$	$1H$	$2H$
$11H$	$11H$	$7H$	$2H$	$1H$

   The identity of G/H is $1H$ and from table we see that the square of every element is $1H$ so there is no element of order 4, and hence G/H is not cyclic. ◀

4. **[15 Points]**

   (a) State Lagrange's Theorem. Be sure to include any hypotheses.

   ▶ **Solution.** If G is a finite group and H is a subgroup, then $|H| \mid |G|$. ◀

   (b) What is the relationship between the order of an element $g \in G$ and the order of the group G?

   ▶ **Solution.** $|g| \mid |G|$. ◀

   (c) If $g \in G$ is an element such that $g \neq 1$ and $g^{20} = g^{30} = 1$, then what can you conclude about $|g|$, the order of g?

   ▶ **Solution.** $g^{20} = 1 \implies |g| \mid 20$ and $g^{30} = 1 \implies |g| \mid 30$. Thus, the order of g is a common divisor of 20 and 30, and hence is a divisor of $\gcd(20, 30) = 10$. Thus, $|g| = 1$, 2, 5, or 10 and 1 is not possible since $g \neq 1$. Thus, $|g| \in \{2, 5, 10\}$. ◀

5. **[10 Points]** Let $\mathbb{R}$ denote the group of real numbers under addition and $\mathbb{R}^*$ denote the group of non-zero real numbers under multiplication. Determine whether each of the given mappings is a homomorphism. Justify your answer briefly.

   (a) Define $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(x) = 3x$ for all $x \in \mathbb{R}$.

▶ **Solution.** The group operation in $\mathbb{R}$ is addition of real numbers and

$$\phi(x+y) = 3(x+y) = 3x + 3y = \phi(x) + \phi(y)$$

so ϕ is a group homomorphism. ◀

(b) Define $\psi : \mathbb{R}^* \to \mathbb{R}^*$ by $\psi(x) = 3x$ for all $x \in \mathbb{R}^*$.

▶ **Solution.** The group operation in $\mathbb{R}^*$ is multiplication of real numbers and $\psi(xy) = 3xy$ while $\psi(x)\psi(y) = (3x)(3y) = 9xy \neq 3xy$ so $\psi(xy) \neq \psi(x)\psi(y)$ and ψ is not a group homomorphism. ◀