Instructions. Answer each of the questions on your own paper and be sure to show your work so that partial credit can be adequately assessed. There is a total of 75 points possible. Put your name on each page of your paper.

1. [**12 Points**] Complete the following definitions.

   (a) A subset I of a ring R is an *ideal* of R if ...

   (b) If R is a commutative ring with identity, and P is an ideal of R with $P \neq R$, then P is a *prime ideal* of R if ...

   (c) If R is a commutative ring with identity, then the *characteristic* of R is ...

2. [**12 Points**] Let $G = \left\{ \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix} : a, b \in \mathbb{R},\ b \neq 0 \right\}$, and $H = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} : a \in \mathbb{R} \right\}$. You may assume that G is a subgroup of the invertible 2×2 matrices with real coefficients under matrix multiplication. Let $\mathbb{R}^*$ denote the group of nonzero real numbers under multiplication.

   (a) Show that the function $f : G \to \mathbb{R}^*$ given by $f\left(\begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}\right) = b$ is a group homomorphism.

   (b) Find $\operatorname{Ker}(f)$.

   (c) Show that H is a normal subgroup of G and that $G/H \cong \mathbb{R}^*$. (*Hint*: Parts (a), (b) and First Isomorphism Theorem.)

3. [**12 Points**]

   (a) Find all distinct isomorphism classes of abelian groups of order 75.

   (b) If G is a finite abelian group of order 75, explain why G has one subgroup of order 5 or six subgroups of order 5.

4. [**12 Points**] Determine which of the following are ideals of the rings given. For those that are, no proof is required. For those which are not, an explanation is required.

   (a) Is $3\mathbb{Z}$ an ideal of $\mathbb{Z}$?

   (b) Is $\mathbb{R}$ an ideal of $\mathbb{R}[x]$?

   (c) Is $I = \left\{ \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} \middle|\ a, c \in \mathbb{Z} \right\}$ an ideal of $M_2(\mathbb{Z})$?

5. **[15 Points]** Let $I = \langle 2 \rangle = 2\mathbb{Z}[i] = \{2a + 2bi : a,\, b \in \mathbb{Z}\}$ be the principal ideal generated by 2 in the Gaussian integers $\mathbb{Z}[i]$.
   (a) Describe all the elements in $\mathbb{Z}[i]/I$, with justification. Give an explicit list of distinct elements.
   (b) Calculate:
      i. $((1 + i) + I) + ((3 - 2i) + I)$
      ii. $((1 + i) + I)((1 + i) + I)$
   (c) Show that I is not a prime ideal in $\mathbb{Z}[i]$.
6. **[12 Points]**Let R be an integral domain.
   (a) Prove that cancellation holds over R. That is, if $a,\, b,\, c \in R$, with $ab = ac$ and $a \neq 0$, then prove that $b = c$.
   (b) An element $c \in R$ is *idempotent* if $c^2 = c$. Prove that in an integral domain R, if c is a nonzero idempotent, then $c = 1$.