Instructions. Answer each of the questions on your own paper. Be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.

1. [**10 Points**] All of the following are commutative rings with identity: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$, and $\mathbb{Z}_6$. Which of these rings are integral domains? Which are fields? You do not need to justify your answers to this question.

2. [**20 Points**] Suppose that R is a commutative ring with identity.
   (a) What properties must a subset I of R satisfy in order to be an ideal?
   (b) Define what it means for an ideal to be *prime*.
   (c) Define what it means for an ideal to be *maximal*.
   (d) Prove that the ideal $I = \{f(x) \in \mathbb{Z}[x] \mid f(0) = 0\}$ of $\mathbb{Z}[x]$ is prime, but not maximal.
   (e) Give an example to show that a factor ring of an integral domain may have zero-divisors. (Recall that a nonzero element a in a ring R is a zero-divisor if there is a nonzero $b \in R$ with $ab = 0$.)

3. [**12 Points**] For the permutation

$$\sigma = \begin{pmatrix}1 & 3 & 5 & 7\end{pmatrix}\begin{pmatrix}2 & 3 & 4 & 7\end{pmatrix}\begin{pmatrix}6 & 1\end{pmatrix}\begin{pmatrix}1 & 7 & 5 & 3\end{pmatrix} \in S_7 :$$

   (a) Write σ as a product of disjoint cycles.
   (b) Compute the order of σ in the group S_7.
   (c) Determine whether σ is even or odd.
   (d) Compute σ^{-1}. You may express your answer in whatever form you wish.

4. [**12 Points**] The alternating group A_4 consists of the 12 elements in S_4 which are even permutations. Let

$$H = \left\{(1), \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3 & 4\end{pmatrix}, \begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}2 & 4\end{pmatrix}, \begin{pmatrix}1 & 4\end{pmatrix}\begin{pmatrix}2 & 3\end{pmatrix}\right\}.$$

   You may assume without proof that H is a subgroup of A_4. The remaining elements of A_4 are the 8 3-cycles in S_4.
   (a) Show that H is isomorphic to the group $\mathbb{Z}_2 \times \mathbb{Z}_2$.
   (b) List the left and right cosets of H in A_4.
   (c) Is H a normal subgroup of A_4?

5. [**10 Points**]

   (a) If $G = \langle a \rangle = \{1, a, a^2, \ldots, a^{11}\}$ is a cyclic group of order 12, then list all of the generators of G.

   (b) The group $U(13)$ of integers modulo 13 with a multiplicative inverse is a cyclic group (under multiplication modulo 13) of order 12. List all of the generators. (*Hint:* $U(13)$ is generated by 2.)

6. [**15 Points**] If G is a group, let $H = \{a \in G \mid a^2 = 1\}$.

   (a) If G is abelian, prove that H is a subgroup of G.

   (b) Give an example of a nonabelian group G such that H is not a subgroup.

7. [**10 Points**] Let $G = \mathbb{Z}_4 \times \mathbb{Z}_2$, let $H = \langle (2, 1) \rangle$ and $K = \langle (2, 0) \rangle$. Show that H is isomorphic to K, but G/H is not isomorphic to G/K.

8. [**10 Points**]

   (a) Compute the greatest common divisor d of the integers 1769 and 2378.

   (b) Write d as a linear combination $d = 1769 \cdot s + 2378 \cdot t$.

9. [**12 Points**]

   (a) Give of list of the distinct isomorphism classes of abelian groups of order 72.

   (b) Which of the groups in your list satisfy the condition $a^{12} = 1$ for all $a \in G$. This condition is written multiplicatively. If the group operation is addition, the condition would be $12a = 0$.

   (c) Which group on your list is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_6$.