Do the following exercises from the text: Chapter 1 (Section 1.3): 7, 15, 22, 24

7. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

   Proof. First show that $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$. For this, let $x \in A \cap (B \cup C)$. This means that $x \in A$ and $x \in B \cup C$. This means that $x \in B$ or $x \in C$. Thus, $x \in A$ and $x \in B$, so that $x \in A \cap B$ or $x \in A$ and $x \in C$ so that $x \in A \cap C$. Therefore, $x \in (A \cap B) \cup (A \cap C)$. Hence $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$.

   Now show that $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$. For this, let $x \in (A \cap B) \cup (A \cap C)$. This means that $x \in A \cap B$ or $x \in A \cap C$.

   Case 1: $x \in A \cap B$. In this case, $x \in A$ and $x \in B$. Since $B \subset B \cup C$, it follows that $x \in A$ and $x \in B \cup C$ and hence $x \in A \cap (B \cup C)$.

   Case 2: $x \in A \cap C$. In this case, $x \in A$ and $x \in C$. Since $C \subset B \cup C$, it follows that $x \in A$ and $x \in B \cup C$ and hence $x \in A \cap (B \cup C)$.

   Combining the two cases gives $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$.

   Since $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$ and $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$, it follows that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. □

15. Prove that $(A \cap B) \setminus (A \cap C) \subset A \cap (B \setminus C)$

   Proof. First show that $A \cap (B \setminus C) \subset (A \cap B) \setminus (A \cap C)$. For this, let $x \in A \cap (B \setminus C)$. Thus $x \in A$ and $x \in B \setminus C$ so that $x \in B$ but $x \notin C$. Therefore, $x \in A \cap B$ but $x \notin A \cap C$ so $x \in (A \cap B) \setminus (A \cap C)$. Hence, $A \cap (B \setminus C) \subset (A \cap B) \setminus (A \cap C)$.

   Now show that $(A \cap B) \setminus (A \cap C) \subset A \cap (B \setminus C)$. For this, let $x \in (A \cap B) \setminus (A \cap C)$. This means that $x \in A \cap B$ but $x \notin A \cap C$. This means that $x \in A$ and $x \in B$ but $x \notin A \cap C$. Since $x \in A$ but $x \notin A \cap C$ it follows that $x \notin C$. Hence, $x \in B \setminus C$ so that, since x is also in A, $x \in A \cap (B \setminus C)$. Therefore, $(A \cap B) \setminus (A \cap C) \subset A \cap (B \setminus C)$.

   Thus, we have shown that $A \cap (B \setminus C) \subset (A \cap B) \setminus (A \cap C)$ and $(A \cap B) \setminus (A \cap C) \subset A \cap (B \setminus C)$ so that $(A \cap B) \setminus (A \cap C) \subset A \cap (B \setminus C)$. □

22. Let $f : A \to B$ and $g : B \to C$ be maps.

   (a) If f and g are both one-to-one functions, show that $g \circ f$ is one-to-one.

   Proof. Assume that f and g are both one-to-one functions, and assume that $g \circ f(a) = g \circ f(b)$ for $a, b \in A$. This implies that $g(f(a)) = g \circ f(a) = g \circ f(b) = g(f(b))$. Since g is one-to-one, this implies that $f(a) = f(b)$ and then, since f is one-to-one, it follows that $a = b$. Thus, we have shown that $g \circ f(a) = g \circ f(b)$ implies that $a = b$, so that $g \circ f$ is one-to-one. □

   (b) If $g \circ f$ is onto, then g is onto.

Proof. Let $c \in C$ be arbitrary. Since $g \circ f$ is onto, then implies that there exists $a \in A$ with $g \circ f(a) = c$. Thus, $g(f(a) = c$. Letting $b = f(a)$ gives an element of B for which $g(b) = g(f(a)) = c$. Since c is arbitrary, this means that g is onto. □

(c) If $g \circ f$ is one-to-one, show that f is one-to-one.

Proof. Suppose that a and b are elements of A with $f(a) = f(b)$. Then $g \circ f(a) = g(f(a)) = g(f(b)) = g \circ f(b)$. Since $g \circ f$ is one-to-one, it follows that $a = b$. Thus, $f(a) = f(b)$ implies that $a = b$, so that f is one-to-one. □

(d) If $g \circ f$ is one-to-one and f is onto, show that g is one-to-one.

Proof. Suppose that $g(b_1) = g(b_2$ for some b_1, $b_2 \in B$. Since f is onto, there are a_1 and a_2 in A with $f(a_1) = b_1$ and $f(a_2) = b_2$. Then $g \circ f(a_1) = g(f(a_1)) = g(b_1) = g(b_2) = g(f(a_2)) = g \circ f(a - 2)$, so that $g \circ f(a_1) = g \circ f(a_2)$. Since $g \circ f$ is one-to-one, it follows that $a_1 = a_2$, which implies that $b_1 = f(a_1) = f(a_2) = b_2$. Therefore, $g(b_1) = g(b_2)$ implies that $b_1 = b_2$, so that g is one-to-one. □

(e) If $g \circ f$ is onto and g is one-to-one, show that f is onto.

Proof. Let $b \in B$ and let $g(b) = c$. By assumption, $g \circ f$ is onto so there is an element $a \in A$ such that $g \circ f(a) = c$. Thus, $g(f(a)) = c$. But $g(b) = c = g(f(a))$ and since g is one-to-one, it follows that $f(a) = b$. Thus, for every $b \in B$, there is an $a \in A$ with $f(a) = b$ and hence f is onto. □

24. Let $f : X \to Y$ be a map with A_1, $A_2 \subset X$ and B_1, $B_2 \subset Y$.

(a) Prove that $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.

Proof. Let $y \in f(A_1 \cup A_2)$. Then there is an $x \in A_1 \cup A_2$ with $y = f(x)$. Then $x \in A_1$ or $x \in A_2$. If $x \in A_1$ then $y = f(x) \in f(A_1) \subset f(A_1) \cup f(A_2)$ and if $x \in A_2$ then $y = f(x) \in f(A_2) \subset f(A_1) \cup f(A_2)$. Thus, $y \in f(A_1 \cup A_2)$ implies that $y \in f(A_1) \cup f(A_2)$, so $f(A_1 \cup A_2) \subset f(A_1) \cup f(A_2)$.

Now let $y \in f(A_1) \cup f(A_2)$. Then $y \in f(A_1)$ or $y \in f(A_2)$. In case $y \in f(A_1)$ there exists some $x_1 \in A_1$ with $y = f(x_1)$. Since $x_1 \in A_1 \subset A_1 \cup A_2$ it follows that $y \in f(A_1 \cup A_2)$. In case $y \in f(A_2)$ there exists some $x_2 \in A_2$ with $y = f(x_2)$. Since $x_2 \in A_2 \subset A_1 \cup A_2$ it follows that $y \in f(A_1 \cup A_2)$. Therefore, $f(A_1) \cup f(A_2) \subset f(A_1 \cup A_2)$.

Since we have shown $f(A_1 \cup A_2) \subset f(A_1) \cup f(A_2)$ and $f(A_1) \cup f(A_2) \subset f(A_1 \cup A_2)$, it follows that $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$. □

(b) Prove that $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$. Give an example in which equality fails.

Proof. Let $y \in f(A_1 \cap A_2)$. This means that there is an $x \in A_1 \cap A_2$ with $y = f(x)$. Since $x \in A_1$ it follows that $y = f(x) \in f(A_1)$ and since $x \in A_2$, it follows that $y = f(x) \in f(A_2)$. Thus, $y \in f(A_1) \cap f(A_2)$ so $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$.

For an example where equality fails let $f : \mathbb{Z} \to \mathbb{Z}$ be the function $f(n) = n^2$. Let $A_1 = \{-1, 0\}$ and $A_2 = \{0, 1\}$. Then, $f(A_1) = \{0, 1\} = A_2 = f(A_2)$ so $f(A_1) \cap f(A_2) = A_2$. But $A_1 \cap A_2 = \{0\}$ so $f(A_1 \cap A_2) = f(\{0\}) = \{0\} \neq f(A_1) \cap f(A_2)$. □

(c) Prove that $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$, where

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

Proof. Let $x \in f^{-1}(B_1 \cap B_2)$. This means that $f(x) \in B_1 \cup B_2$, so $f(x) \in B_1$ or $f(x) \in B_2$. That is $x \in f^{-1}(B_1)$ or $x \in f^{-1}(B_2)$ so that $x \in f^{-1}(B_1) \cup f^{-1}(B_2)$. Hence, $f^{-1}(B_1 \cup B_2) \subset f^{-1}(B_1) \cup f^{-1}(B_2)$.

Now assume that $x \in f^{-1}(B_1) \cup f^{-1}(B_2)$. Then $x \in f^{-1}(B_1)$ or $x \in f^{-1}(B_2)$. Thus, $f(x) \in B_1$‘ or $f(x) \in B_2$. That is, $f(x) \in B_1 \cup B_2$ so $x \in f^{-1}(B_1 \cup B_2)$. Hence, $f^{-1}(B_1) \cup f^{-1}(B_2) \subset f^{-1}(B_1 \cup B_2)$.

These two inclusions show that $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$. □

(d) Prove that $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

Proof. Let $x \in f^{-1}(B_1 \cap B_2)$. This means that $f(x) \in B_1 \cap B_2$ so that $f(x) \in B_1$ and $f(x) \in B_2$. Therefore, $x \in f^{-1}(B_1)$ and $x \in f^{-1}(B_2)$, which means $x \in f^{-1}(B_1) \cap f^{-1}(B_2)$. Hence we have shown that $f^{-1}(B_1 \cap B_2) \subset f^{-1}(B_1) \cap f^{-1}(B_2)$.

Now assume that $x \in f^{-1}(B_1) \cap f^{-1}(B_2)$. Then $x \in f^{-1}(B_1)$ and $x \in f^{-1}(B_2)$. Thus, $f(x) \in B_1$‘ and $f(x) \in B_2$. That is, $f(x) \in B_1 \cap B_2$ so $x \in f^{-1}(B_1 \cap B_2)$. Hence, $f^{-1}(B_1) \cap f^{-1}(B_2) \subset f^{-1}(B_1 \cap B_2)$.

These two inclusions show that $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$. □

(e) Prove that $f^{-1}(Y \setminus B_1) = X \setminus f^{-1}(B_1)$.

Proof.

$$\begin{aligned} x \in f^{-1}(Y \setminus B_1) &\iff f(x) \in Y \setminus B_1 \\ &\iff f(x) \notin B_1 \\ &\iff x \notin f^{-1}(B_1) \\ &\iff x \in X \setminus f^{-1}(B_1). \end{aligned}$$

□

Supplemental Exercises

1. In each case, state whether the mapping is onto, one-to-one, or bijective. Justify your answer.

   (a) $f : \mathbb{Z} \times \mathbb{Z}^+ \to \mathbb{Q}$ defined by $f(n, m) = \frac{n}{m}$.

▶ **Solution.** f is not one-to-one since $f(1,\,2) = \frac{1}{2} = \frac{2}{4} = f(2,\,4)$ and $(1,\,2) \neq (2,\,4)$. f is onto since any rational number can be written as $\frac{n}{m}$ where m can be taken to be positive since $\frac{n}{m} = \frac{-n}{-m}$. Then $\frac{n}{m} = f(n,\,m)$. f is not bijective since it is not one-to-one (bijective means both one-to-one and onto. ◀

(b) $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even.} \end{cases}$

▶ **Solution.** f is not one-to-one since $f(1) = (1+1)/2 = 1 = 2/2 = f(2)$ and $1 \neq 2$. f is onto since $n = f(2n)$ for every $n \in \mathbb{N}$. f is not bijective since it is not one-to-one. ◀

2. In each case, decide whether the relation $\equiv$ is an equivalence relation on A. Give reasons for your answer. If it is an equivalence relation, describe the equivalence classes.

(a) $A = \{-1,\,0,\,1\}$; $a \equiv b$ if $a^2 = b^2$.

▶ **Solution.** This is an equivalence relation. To see this, check the 3 defining properties: $a \equiv a$ since $a^2 = a^2$ so the reflexive property holds. If $a \equiv b$ then $a^2 = b^2$, which implies that $b^2 = a^2$ so that $b \equiv a$. Thus, the symmetric property holds. If $a \equiv b$ and $b \equiv c$ then $a^2 = b^2$ and $b^2 = c^2$, so by transitivity of equality, $a^2 = c^2$ so that $a \equiv c$. Therefore, the transitive property also holds.

The equivalence classes are $[-1] = \{-1,\,1\} = [1]$ and $[0] = \{0\}$. ◀

(b) $A = \mathbb{N}$; $a \equiv b$ if $a \leq b$.

▶ **Solution.** This is not an equivalence relation since the symmetric property fails. To see this, note that $0 \equiv 1$ since $0 \leq 1$, but $1 \not\leq 0$ so $1 \not\equiv 0$. ◀

(f) $A =$ the set of all subsets of $\{1,\,2,\,3\}$; $X \equiv Y$ if $|X| = |Y|$.

▶ **Solution.** This is an equivalence relation. To see this, check the 3 defining properties: $X \equiv X$ for all $X \subseteq \{1,\,2,\,3\}$ since $|X| = |X|$. Thus the reflexive property holds. If $X \equiv Y$ then $|X| = |Y|$ so $|Y| = |X|$ and hence $Y \equiv X$. Thus, the symmetric property holds. If $X \equiv Y$ and $Y \equiv Z$ then $|X| = |Y|$ and $|Y| = |Z|$. Thus, $|X| = |Z|$ so that $X \equiv Z$ and the transitive property holds.

The equivalence classes are:

$$
\begin{aligned}
[\phi] &= \{\phi\} \\
[\{1\}] &= \{\{1\}\,,\,\{2\}\,,\,\{3\}\} = [\{2\}] = [\{3\}] \\
[\{1,\,2\}] &= \{\{1,\,2\}\,,\,\{1,\,3\}\,,\,\{2,\,3\}\} = [\{1,\,3\}] = [\{2,\,3\}] \\
[\{1,\,2,\,3\}] &= \{\{1,\,2,\,3\}\}
\end{aligned}
$$

◀