Do the following exercises from Judson:
Chapter 5, Section 5.3: 7, 14, 30, 34

7. Find all possible orders of elements in S_7 and A_7.

   ► **Solution.** To find the order of σ, write σ in disjoint cycle format. The order is then the least common multiple of the lengths of the cycles appearing in the disjoint cycle factorization. If $\sigma \neq \varepsilon$, then σ can be written as a product $\sigma = \gamma_1\gamma_2\ldots\gamma_k$ of disjoint cycles of length ≥ 2. Since each cycle uses at least 2 of the 7 elements of $\{1, 2, \ldots, n\}$ we must have $k \leq 3$. If $k = 1$, then σ is a cycle of length ≤ 7 so the possible orders are 2, 3, 4, 5, 6, 7. If $k = 2$ then possible cycle structures are $\gamma_1\gamma_2$ where (a) γ_1 is a 2-cycle and γ_2 is an r-cycle for $2 \leq r \leq 5$, which gives possible orders for σ of 2, 6, 4, 10, (b) γ_1 is a 3 cycle and γ_2 is an r-cycle for $3 \leq r \leq 4$, which gives possible orders for σ of 3 and 12. If γ_1 has length greater than 3, then the cycle structure is already covered by one of the above cases, so no new possible orders occur. If $k = 2$ then $\sigma = \gamma_1\gamma_2\gamma_3$ and the only possibilities are all are 2-cycles or 2 are 2-cycles and 1 is a 3-cycle. This gives orders 2 and 6. Thus, the possible orders of elements of S_7 are 1, 2, 3, 4, 5, 6, 7, 10, 12.

   Since A_7 is a subgroup of S_7, to find the possible orders of elements of A_7, just pick out the possible cycle structures in S_7 which correspond to even permutations. Since

   $$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \end{pmatrix} = \begin{pmatrix} a_1 & a_r \end{pmatrix} \begin{pmatrix} a_1 & a_{r-1} \end{pmatrix} \cdots \begin{pmatrix} a_1 & a_3 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \end{pmatrix},$$

   an r-cycle is even if and only if r is odd. Thus, in writing a permutation σ in disjoint cycle form, σ is even if and only if there are an even number of cycles of even length. Thus, a single r cycle in A_7 must have order 3, 5, or 7. If σ is a product of 2 cycles γ_1 and γ_2, then both must be of odd order, or both must be of even order. Both of odd order would require that both are 3-cycles, and thus the order would be 3. If both are of even order, then σ is a product of 2 disjoint 2-cycles, or a product of a 2-cycle and a 4-cycle. Thus, σ would have order 2 or 4. The only product of 3 disjoint cycles in A_7 is a product of two 2-cycles and a 3-cycle, which is odd and thus not in A_7. Thus, the possible orders of elements in A_7 are 1, 2, 3, 4, 5, 6, 7 ◄

14. Using cycle notation, list the elements in D_5.

▶ **Solution.**

$$\begin{aligned}
\varepsilon &= (1) \\
r &= \begin{pmatrix}1 & 2 & 3 & 4 & 5\end{pmatrix} \\
r^2 &= \begin{pmatrix}1 & 3 & 5 & 2 & 4\end{pmatrix} \\
r^3 &= \begin{pmatrix}1 & 4 & 2 & 5 & 3\end{pmatrix} \\
r^4 &= \begin{pmatrix}1 & 5 & 4 & 3 & 2\end{pmatrix} \\
s &= \begin{pmatrix}2 & 5\end{pmatrix}\begin{pmatrix}3 & 4\end{pmatrix} \\
sr &= \begin{pmatrix}1 & 5\end{pmatrix}\begin{pmatrix}2 & 4\end{pmatrix} \\
sr^2 &= \begin{pmatrix}1 & 4\end{pmatrix}\begin{pmatrix}2 & 3\end{pmatrix} \\
sr^3 &= \begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}4 & 5\end{pmatrix} \\
sr^4 &= \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3 & 5\end{pmatrix}
\end{aligned}$$

◀

30. Let $\tau = \begin{pmatrix}a_1 & a_2 & \cdots & a_k\end{pmatrix}$ be a cycle of length k.

   (a) Prove that if σ is any permutation, then

$$\sigma\tau\sigma^{-1} = \begin{pmatrix}\sigma(a_1) & \sigma(a_2) & \cdots & \sigma(a_k)\end{pmatrix}$$

▶ **Solution.** If $j \neq \{\sigma(a_1), \cdots, \sigma(a_k)\}$ then $\sigma^{-1}(j) \neq \{a_1, \cdots, a_k\}$ so $\sigma^{-1}(j)$ is fixed by τ so that $\sigma\tau\sigma^{-1}(j) = \sigma(\tau(\sigma^{-1}(j))) = \sigma(\sigma^{-1}(j)) = j$. Also, if $1 \leq i < k$

$$\sigma\tau\sigma^{-1}(\sigma(a_i)) = \sigma\tau(\sigma^{-1}(\sigma(a_i))) = \sigma\tau(a_i) = a_{i+1},$$

and if $i = k$, then

$$\sigma\tau\sigma^{-1}(\sigma(a_k)) = \sigma\tau(\sigma^{-1}(\sigma(a_k))) = \sigma\tau(a_k) = a_1,$$

Thus, $\sigma\tau\sigma^{-1}$ and $\begin{pmatrix}\sigma(a_1) & \sigma(a_2) & \cdots & \sigma(a_k)\end{pmatrix}$ are the same permutation. ◀

   (b) Let $\mu = \begin{pmatrix}b_1 & b_2 & \cdots & b_k\end{pmatrix}$ be any cycle of length k. Prove that there is a permutation σ such that $\sigma\tau\sigma^{-1} = \mu$.

▶ **Solution.** Define the permutation σ as follows. σ is a permutation of $\{1, 2, \cdots, n\}$ such that $\sigma(a_i) = b_i$ for $1 \leq i \leq k$. The sets $A = \{1, 2, \cdots, n\} \setminus \{a_1, \cdots, a_k\}$ and $B = \{1, 2, \cdots, n\} \setminus \{b_1, \cdots, b_k\}$ both have $n-k$ elements. Thus there is a one-to-one correspondence $f : A \to B$. If $j \in A$, define $\sigma(j) = f(j)$. Then σ is a permutation in S_n and $\sigma\tau\sigma^{-1} = \mu$ by part (a). ◀

34. If α is even, prove that α^{-1} is also even.

▶ **Solution.** Write α as a product of transpositions $\alpha = \tau_1\tau_2\cdots\tau_m$. Then,

$$\begin{aligned}\alpha^{-1} &= \tau_n^{-1}\tau_{n-1}^{-1}\cdots\tau_1^{-1}\\ &= \tau_n\tau_{n-1}\cdots\tau_1\end{aligned}$$

Since the inverse of a transposition is itself. Thus, if α can be written as a product of m transpositions, then α^{-1} can be written as the same number of transpositions. Hence, α is even if and only if α^{-1} is even. ◀

Chapter 6, Section 6.4: 16, 17

16. If $|G| = 2n$, prove that the number of elements of order 2 is odd. Use this to show that G must contain a subgroup of order 2.

    ▶ **Solution.** Divide G into three sets: $A = \{1\}$, B is the set of elements of order 2, and C is the set of elements of order > 2. If $g = g^{-1}$ then $g^2 = 1$ so the order of g is 2 or 1, so $g \in A \cup B$. If the order of g is greater than 2, then $g \neq g^{-1}$ and of course g^{-1} also has order greater than 2. Thus, the elements of C can be divided up into sets of 2 by takin $\{g, g^{-1}\}$. All of these sets have 2 elements since $g \neq g^{-1}$. Thus, $|C| = 2k$ is even. Since $G = A \cup B \cup C$ and these sets are mutually disjoint, it follows that $2n = |G| = |A| + |B| + |C| = 1 + |B| + 2k$ so $|B| = 2(n - k) - 1$ is odd. Since the number of elements of order 2 is odd, there must be at least 1 such element. If $a \in G$ is an element of order 2, then $H = \{1, a\}$ is a subgroup of order 2. ◀

1. Suppose that $[G : H] = 2$. If a and b are not in H, show that $ab \in H$.

    ▶ **Solution.** Suppose $a \notin H$. Since $[G : H] = 2$ there are two cosets. Since $a \notin H$, a^{-1} is also not in H and the two cosets are H and $a^{-1}H$. But if b is also not in H, then b must be in the other coset, namely $a{-}1H$. Therefore, $b = a^{-1}h$ for some $h \in H$ and hence $ab = h \in H$. ◀

Exercises not from the text:

1. Let $\sigma = \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9\\ 3 & 8 & 9 & 5 & 2 & 1 & 6 & 4 & 7\end{pmatrix}$ and $\tau = \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9\\ 2 & 8 & 6 & 9 & 4 & 7 & 3 & 1 & 5\end{pmatrix}$. For each of the permutations σ, τ, σ^2, $\sigma\tau$, and $\tau\sigma$, determine the following.

    (a) Find the disjoint cycle factorization of each permutation.

    ▶ **Solution.**

    $$\begin{aligned}\sigma &= \begin{pmatrix}1 & 3 & 9 & 7 & 6\end{pmatrix}\begin{pmatrix}2 & 8 & 4 & 5\end{pmatrix} & \tau &= \begin{pmatrix}1 & 2 & 8\end{pmatrix}\begin{pmatrix}3 & 6 & 7\end{pmatrix}\begin{pmatrix}4 & 9 & 5\end{pmatrix}\\ \sigma^2 &= \begin{pmatrix}1 & 9 & 6 & 3 & 7\end{pmatrix}\begin{pmatrix}2 & 4\end{pmatrix}\begin{pmatrix}5 & 8\end{pmatrix} & \sigma\tau &= \begin{pmatrix}1 & 8 & 3\end{pmatrix}\begin{pmatrix}2 & 4 & 7 & 9\end{pmatrix}\\ \tau\sigma &= \begin{pmatrix}1 & 6 & 2\end{pmatrix}\begin{pmatrix}3 & 5 & 8 & 9\end{pmatrix}\end{aligned}$$

    ◀

(b) Compute the order of each permutation.

▶ **Solution.** $|\sigma| = \text{lcm}(5, 4) = 20$, $|\tau| = \text{lcm}(3, 3, 3) = 3$, $|\sigma^2| = \text{lcm}(5, 2, 2) = 10$, $|\sigma\tau| = \text{lcm}(3, 4) = 12$, $|\tau\sigma| = \text{lcm}(3, 4) = 12$ ◀

(c) Determine if each permutation is even or odd.

▶ **Solution.** A cycle is even if and only if the length if odd and the product of cycles is even if and only if there are an even number of even length cycles. thus, σ is a product of an even and an odd cycle, and hence is odd. τ is a product of odd order and hence even cycles. Thus $\tau is even.$ The square of any permutation is even, so σ^2 is even. $\sigma\tau$ and $\tau\sigma$ are each a product of an odd and an even cycle, hence both are odd permutations. ◀

2. Let $G = D_4 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$ where $r^4 = 1$, $s^2 = 1$, $srs = r^{-1}$. (See Theorem 5.23) A representation as permutations is $r = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$ and $s = \begin{pmatrix} 2 & 4 \end{pmatrix}$. Let $H = \langle r^2 \rangle$ and $K = \langle s \rangle$. Determine all of the left and right cosets of the subgroups H and K of G.

▶ **Solution.** Note that $srs = r^{-1}$ implies $sr^k = r^{-k}s$ so $sr^k = r^{-k}s$. Then

$$\begin{aligned} 1H &= \left\{1, r^2\right\} = r^2H & \qquad H1 &= \left\{1, r^2\right\} = Hr^2 \\ rH &= \left\{r, r^3\right\} = r^3H & Hr &= \left\{r, r^3\right\} = Hr^3 \\ sH &= \left\{s, sr^2\right\} = sr^2H & Hs &= \left\{s, sr^2\right\} = Hs \\ srH &= \left\{sr, sr^3\right\} = sr^3H & Hsr &= \left\{sr, sr^3\right\} = Hsr \end{aligned}$$

and

$$\begin{aligned} 1K &= \{1, s\} = sK & \qquad K1 &= \{1, s\} = Ks \\ rK &= \left\{r, rs = sr^3\right\} = sr^3K & Kr &= \{r, sr\} = Ksr \\ r^2K &= \left\{r, r^2s = sr^2\right\} = sr^2K & Kr^2 &= \left\{r^2, sr^2\right\} \\ r^3K &= \left\{r^3, r^3s = sr\right\} = srK & Kr^3 &= \left\{r^3, sr^3\right\} \end{aligned}$$

◀

3. Let G be a group and let $g \in G$ be an element. Suppose $|G| = 40$, $g^8 \neq 1$, and $g^{20} \neq 1$. Show that $G = \langle g \rangle$.

▶ **Solution.** The order of g divides the order of G, which is 40, so $|g| \mid 40$. Since $g^8 \neq 1$, the order of g cannot be 1, 2, 4, or 8. Since $g^{20} \neq 1$, the order of g cannot be 1, 4, 5, 10 or 20. The only remaining divisor of 40 is 40, so $|g| = 40$ and so $G = \langle g \rangle$. ◀

4. Suppose a group G has subgroups of orders 45 and 75. If $|G| < 400$ determine $|G|$.

▶ **Solution.** By Lagrange's theorem, both 45 and 75 must divide $|G|$. Thus, $|G|$ is a common multiple of 45 and 75 and hence $|G|$ is a multiple of the least common multiple of 45 and 75, which is 225. Since $|G| < 400$, the only possibility is $|G| = 225$. ◀