Do the following exercises from Judson:
Chapter 9, Section 9.3: 5, 11, 12, 16 (d)

5. Show that $U(5)$ is isomorphic to $U(10)$, but $U(12)$ is not isomorphic to $U(10)$.

   ► **Solution.** $U(5) = \{2, 2^2 = 4, 2^3 = 3, 2^4 = 1\} = \langle 2 \rangle$ is cyclic of order 4 with generator 2. $U(10) = \{3, 3^2 = 9, 3^3 = 7, 3^4 = 1\} = \langle 3 \rangle$ is cyclic of order 4 with generator 3. Thus, there is an isomorphism $\phi : U(5) \to U(10)$ given by $\phi(2^k) = 3^k$ for all $k \in \mathbb{Z}$. This is well defined since $2^4 = 1 \in U(5)$ while $3^4 = 1 \in U(10)$. Since there are only 4 elements, this amounts to $\phi(1) = 1$, $\phi(2) = 3$, $\phi(4) = 9$, and $\phi(3) = 7$, which is clearly one-to-one and onto. Since $\phi(2^k 2^m) = \phi(2^{k+m}) = 3^{k+m} = 3^k 3^m$, it follows that ϕ is a group homomorphism.

   $U(12) = \{1, 5, 7, 11\}$. Since $2^2 = 1$, $5^2 = 1$, and $11^2 = 1$, all elements other than the identity have order 2. Since there is no element of order 4, $U(12)$ cannot be isomorphic to $U(10)$, which has an element, namely 3, of order 2. ◄

11. Find five non-isomorphic groups of order 8.

    ► **Solution.** The five groups are $\mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, D_4 (the symmetry group of a square), Q_8 (the quaternion group). The first 3 are abelian, so none of the first 3 are isomorphic to D_4 or Q_8, since these are both non-abelian. D_4 has 2 elements of order 4, namely r and r^3, where r is the rotation by $90°$. Q_8 has 6 elements of order 4, namely $\pm i$, $\pm j$, $\pm k$. Thus D_4 is not isomorphic to Q_8. $\mathbb{Z}_8$ has an element of order 8, namely 1, $\mathbb{Z}_2 \times \mathbb{Z}_4$ has an element of order 4, but no element of higher order, and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ has all non-identity elements of order 2. Thus, none of the three abelian groups are isomorphic to another of the three. Hence, all of these groups of order 8 are distinct, in that none is isomorphic to another of the listed groups of order 8. ◄

12. Prove that S_4 is not isomorphic to D_{12}.

    ► **Solution.** Both groups have order 24 and are non-abelian. However, D_{12} has an element of order 12, namely, the rotation r by $30°$, whereas, the largest order of an element in S_4 is 4, given by a 4-cycle. Since S_4 does not have an element of order 12, it cannot be isomorphic to D_{12}. ◄

1. Find the order of the element $(8, 8, , 8)$ in $\mathbb{Z}_{10} \times \mathbb{Z}_{24} \times \mathbb{Z}_{80}$.

   ► **Solution.** In $\mathbb{Z}_{10}$ the order of 8 is $10/\gcd(8, 10) = 10/2 = 5$. In $\mathbb{Z}_{24}$ the order of 8 is $24/\gcd(8, 24) = 24/8 = 3$, while in $\mathbb{Z}_{80}$ the order of 8 is $80/\gcd(8, 80) = 80/8 = 10$. Thus, the order of $(8, 8, , 8)$ in $\mathbb{Z}_{10} \times \mathbb{Z}_{24} \times \mathbb{Z}_{80}$ is the least common multiple of 5, 3, 10, which is 30. ◄

Chapter 10, Section 10.3: 4, 7

4. Let T be the group of nonsingular upper triangular 2×2 matrices with entries in $\mathbb{R}$; that is matrices of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ where a, b, $c \in \mathbb{R}$ and $ac \neq 0$. Let U consist of matrices of the form $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$ where $x \in \mathbb{R}$.

   (a) Show that U is a subgroup of T.

   ▶ **Solution.** First, $U \subset T$ is clear since each element of U is upper triangular and nonsingular (each has determinant 1). The identity matrix is in U, and

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x+y \\ 0 & 1 \end{bmatrix} \in U,$$

   while $\begin{bmatrix} 1 & x \\ 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -x \\ 0 & 0 \end{bmatrix}$. Thus, U contains the identity of T, is closed under matrix multiplication, and is closed under inverses. Thus U is a subgroup of T. ◀

   (b) Prove that U is abelian.

   ▶ **Solution.**

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x+y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & y+x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

   Thus, multiplication in U is commutative and U is abelian. ◀

   (c) Prove that U is a normal subgroup of T.

   ▶ **Solution.** Let $A = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \in U$ and let $B = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in T$ be arbitrary. To show that U is normal in T it is sufficient to show that $BAB^{-1} \in U$ for al choices of $A \in U$ and $B \in T$. (By Theorem 10.3.)

$$\begin{aligned} BAB^{-1} &= \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{a} & -\frac{b}{ac} \\ 0 & \frac{1}{c} \end{bmatrix} \\ &= \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} \frac{1}{a} & -\frac{b}{ac} + \frac{x}{c} \\ 0 & \frac{1}{c} \end{bmatrix} \\ &= \begin{bmatrix} 1 & -\frac{b}{c} + \frac{xa}{c} + \frac{b}{c} \\ 0 & 1 \end{bmatrix} \end{aligned}$$

   The last matrix is in U for all choices of A and B, so U is normal in T. ◀

   (d) Show that the factor group T/U is abelian.

► **Solution.** The elements of T/U are the cosets AU where $A \in T$, and multiplication in T/U is coset multiplication, that is, $(AU)(BU) = (AB)U$. Thus, to show that T/U is abelian, it is necessary to show that $(AB)U = (BA)U$ for all A, $B \in T$. The condition for the two left cosets to be equal is (from Lemma 6.3 (5)): $(AB)^{-1}(BA) \in U$. Thus, we need to show that $B^{-1}A^{-1}BA \in U$ for all A, $B \in T$. If $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ and $B = \begin{bmatrix} r & s \\ 0 & t \end{bmatrix}$, then

$$\begin{aligned} B^{-1}A^{-1}BA &= \begin{bmatrix} \frac{1}{r} & -\frac{s}{rt} \\ 0 & \frac{1}{t} \end{bmatrix} \begin{bmatrix} \frac{1}{a} & -\frac{b}{ac} \\ 0 & \frac{1}{c} \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} r & s \\ 0 & t \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{ra} & -\frac{b}{rac} - \frac{s}{rct} \\ 0 & \frac{1}{tc} \end{bmatrix} \begin{bmatrix} ar & rb + cs \\ 0 & ct \end{bmatrix} \\ &= \begin{bmatrix} 1 & \frac{b}{a} + \frac{cs}{ra} - \frac{bt}{ra} - \frac{s}{r} \\ 0 & 1 \end{bmatrix} \in U \end{aligned}$$

Thus $(AB)U = (BA)U$ so T/U is abelian. ◄

(e) Is T normal in $\mathrm{GL}_2(\mathbb{R})$?

► **Solution.** If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $A = A^{-1}$ and $B \in T$. Then,

$$\begin{aligned} ABA^{-1} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \notin T. \end{aligned}$$

Thus T is not a normal subgroup of $\mathrm{GL}_2(\mathbb{R})$. Note that since the above B is also in U, we have also shown that U is not a normal subgroup of $\mathrm{GL}_2(\mathbb{R})$. ◄

1. Prove or disprove: If H is a normal subgroup of G such that H and G/H are abelian, then G is abelian.

► **Solution.** This is false. One counterexample is provided by Exercise 4. Namely, the subgroup U of T is an abelian and normal subgroup of T and T/U is abelian, but the group T is not abelian. For example, if $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ then

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = BA$$

Another example is to take $G = S_3$ and $H = \langle \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \rangle$. Then H is a cyclic subgroup of G of order 3, so the index $[G : H] = 2$ and hence H is a normal subgroup. The factor group G/H has order $6/3 = 2$. Since 2 is prime this means that G/H is cyclic of order 2, and hence abelian. Thus, both H and G/H are abelian, but $G = S_3$ is not abelian. ◄

Exercises not from the text:

1. If G is any group, define $\alpha : G \to G$ by $\alpha(g) = g^{-1}$. Show that G is abelian if and only if α is a homomorphism.

   ▶ **Solution.** Assume that G is abelian. then

   $$\alpha(gh) = (gh)^{-1} = h^{-1}g^{-1} = g^{-1}h^{-1} = \alpha(g)\alpha(h)$$

   where the third equality is because G is abelian. Thus, α is a group homomorphism.

   Conversely, assume that α is a group homomorphism and let g, $h \in G$. Then

   $$\begin{aligned} gh &= (g^{-1})^{-1}(h^{-1})^{-1} \\ &= \alpha(g^{-1})\alpha(h^{-1}) \\ &= \alpha(g^{-1}h^{-1}) \\ &= \alpha((hg)^{-1}) \\ &= ((hg)^{-1})^{-1} \\ &= hg \end{aligned}$$

   where the third equality is because α is a homomorphism. Since g and h are arbitrary, it follows that G is abelian. ◀

2. If $\alpha : G \to G_1$ is a homomorphism, show that $K = \{g \in G : \alpha(g) = 1\}$ is a subgroup of G.

   ▶ **Solution.** The identity 1 of G is in K since $\alpha(1) = 1$ for any homomorphism. If g, $h \in K$, then $\alpha(gh) = \alpha(g)\alpha(h) = 1 \cdot 1 = 1$ so $gh \in K$. If $g \in K$ then $\alpha(g^{-1}) = \alpha(g)^{-1} = 1^{-1} = 1$ so $g^{-1} \in K$. Thus K is a subgroup of G. ◀

3. In each case determine whether $\alpha : G \to G_1$ is an isomorphism.

   (a) $G = G_1 = \mathbb{Z}$, $\alpha(n) = 2n$.

   ▶ **Solution.** α is a homomorphism, but it is not an isomorphism since it is not onto. In fact $\alpha(G) = 2\mathbb{Z} \neq \mathbb{Z}$. ◀

   (b) $G = G_1 = \mathbb{Z}_5^*$, $\alpha(g) = g^3$.

   ▶ **Solution.** This is a homomorphism since $\mathbb{Z}_5^*$ is abelian so $\alpha(gh) = (gh)^3 = g^3h^3 = \alpha(g)\alpha(h)$. Since $0^3 = 0$, $1^3 = 1$, $2^3 = 3$, $3^3 = 2$, and $4^3 = 4$ in $\mathbb{Z}_5^*$, it follows that α is bijective, and hence an isomorphism. ◀

   (c) $G = G_1 = \mathbb{Z}_8$, $\alpha(g) = 2g$.

   ▶ **Solution.** This is a homomorphism since $\alpha(g + h) = 2(g + h) = 2g + 2h = \alpha(g) + \alpha(h)$. It is not an isomorphism since it is not injective: $\alpha(0) = 0 = \alpha(4)$ but $4 \neq 0$ in $\mathbb{Z}_8$. ◀