Do the following exercises from Judson:
Chapter 11, Section 11.3: 7, 13

7. In the group $\mathbb{Z}_{24}$, let $H = \langle 4 \rangle$ and : $N = \langle 6 \rangle$.

   (a) List the elements in $H + N$ and $H \cap N$.

   ▶ **Solution.** $H = \{0, 4, 8, 12, 16, 20\}$ and $N = \{0, 6, 12, 18\}$. Thus,

   $$H + N = \{0, 2, 4, 6, 8, 10, 12, 12, 14, 16, 18, 20, 22\} = \langle 2 \rangle.$$

   and $H \cap N = \{0, 12\}$. ◀

   (b) List the cosets in $(H + N)/N$.

   ▶ **Solution.** $0 + N = N = \{0, 6, 12, 18\}$, $2 + N = \{2, 8, 14, 20\}$, $4 + N = \{4, 10, 16, 22\}$. ◀

   (c) List the cosets in $H/(H \cap N)$.

   ▶ **Solution.** $0+(H\cap N) = H\cap N = \{0, 12\}$, $4+(H\cap N) = \{4, 16\}$, $8+(H\cap N) = \{8, 20\}$. ◀

   (d) Give the correspondence between $(H + N)/N$ and $H/(H \cap N)$ described in the proof of the Second Isomorphism Theorem.

   ▶ **Solution.** The correspondence is $\phi(h + (H \cap N)) = h + N$. This gives: $\phi(0 + (H\cap N)) = 0+N$, $\phi(4+(H\cap N)) = 4+N$, and $\phi(8+(H\cap N)) = 8+N = 2+N$. ◀

13. Prove or disprove: $\mathbb{Q}/\mathbb{Z} \cong \mathbb{Q}$.

    ▶ **Solution.** Note that every nonzero element of $\mathbb{Q}$ has infinite order in the additive group $\mathbb{Q}$. To see this, suppose that $q \neq 0 \in \mathbb{Q}$. In an additive group the k^{th} power of q is kq and $kq = 0$ if and only if $k = 0$. Thus, there is no element of finite order in $\mathbb{Q}$. However, in the group $\mathbb{Q}/\mathbb{Z}$, every element has finite order. To see this, suppose that $q \in \mathbb{Q}$ and consider the coset $q + \mathbb{Z}$. Since $q \in \mathbb{Q}$, $q = \frac{r}{s}$ where r, $s \in \mathbb{Z}$ with $s \neq 0$. Then $s(q + \mathbb{Z}) = sq + \mathbb{Z} = r + \mathbb{Z} = 0 + \mathbb{Z}$ since $r \in \mathbb{Z}$. Thus, $q + \mathbb{Z}$ has finite order ($\leq s$) in $\mathbb{Q}/\mathbb{Z}$. Since $\mathbb{Q}/\mathbb{Z}$ has nonzero elements of finite order and $\mathbb{Q}$ does not, they cannot be isomorphic. Any isomorphism preserves the order of elements. ◀

Exercises not from the text:

1. Let $T = \{z \in \mathbb{C}^* \mid |z| = 1\}$. Prove that $\mathbb{C}^*/T$ is isomorphic to $\mathbb{R}^+$, the group of positive real numbers under multiplication.

   ▶ **Solution.** Define $\phi : \mathbb{C}^* \to \mathbb{R}^+$ by $\phi(z) = |z|$. Since $\phi(zw) = |zw| = |z|\,|w| = \phi(z)\phi(w)$, ϕ is a group homomorphism. Moreover, $\mathrm{Ker}(\phi) = \{z \in \mathbb{C}^* : \phi(z) = |z| = 1\} = T$. ϕ is onto since $\phi(a) = a$ for all $a \in \mathbb{R}^+$. By the first isomorphism theorem, $\mathbb{C}^*/T = \mathbb{C}^*/\operatorname{Ker}(\phi) \cong \phi(\mathbb{C}^*) = \mathbb{R}^+$. ◀

2. Suppose that G is a finite group and that $\phi : G \to \mathbb{Z}_{10}$ is an onto group homomorphism. What can we say about the order of G? Generalize this statement.

   ▶ **Solution.** By the first isomorphism theorem, $G/\operatorname{Ker}(\phi) \cong \mathbb{Z}_10$. Thus, by Lagrange's theorem, $|G| / |\operatorname{Ker}(\phi)| = |\mathbb{Z}_{10}| = 10$. Therefore, 10 divides the order of G.

   In general, if $\phi : G \to H$ is onto, then the order of H divides the order of G. ◀

3. If H and K are normal subgroups of a group G and $H \cap K = \{e\}$, prove that G is isomorphic to a subgroup of $G/H \times G/K$.

   ▶ **Solution.** Define $\phi : G \to G/H \times G/K$ by $\phi(g) = (gH, gK)$. Then ϕ is a homomorphism since

$$\begin{aligned}\phi(g_1g_2) &= (g_1g_2H,\, g_1g_2K) \\ &= ((g_1H)(g_2H),\, (g_1K)(g_2K) \\ &= (g_1H,\, g_1K)(g_2H,\, g_2K) \\ &= \phi(g_1)\phi(g_2).\end{aligned}$$

   Then by the first isomorphism theorem $G/\operatorname{Ker}(\phi)$ is isomorphic to $\phi(G) \subset G/H \times G/K$. But $g \in \operatorname{Ker}(\phi)$ if and only if $\phi(g) = (gH, gK) = (H, K)$ if and only if $g \in H$ and $g \in K$, i.e., $g \in H \cap K = \{e\}$. Hence $\operatorname{Ker}(\phi) = \{e\}$ and $G \cong \phi(G)$, which is a subgroup of $G/H \times G/K$. ◀

4. Let N be a normal subgroup of a finite group G. Prove that the order of the group element gN in G/N divides the order of g.

   ▶ **Solution.** Let n be the order of g. Then $g^n = e$ and $(gN)^n = g^nN = eN = N$, which is the identity in G/N. Thus, the order of gN divides n. ◀