1. Calculate the number of elements of order 2 in each of the abelian groups $\mathbb{Z}_{16}$, $\mathbb{Z}_8 \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_4$, and $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Do the same for elements of order 4.

   ► **Solution.**
   - $\mathbb{Z}_{16}$:
     A cyclic group has a unique subgroup of order dividing the order of the group. Thus, $\mathbb{Z}_{16}$ has one subgroup of order 2, namely $\langle 8 \rangle$, which gives the only element of order 2, namely 8. There is one subgroup of order 4, namely $\langle 4 \rangle$, and this subgroup has 2 generators, each of order 4. Thus the 2 elements of order 4 in $\mathbb{Z}_{16}$ are 4 and 12.
   - $\mathbb{Z}_8 \times \mathbb{Z}_2$:
     The order of (r, s) is the least common multiple of the order of r and s. Thus elements of order 2 in $\mathbb{Z}_8 \times \mathbb{Z}_2$ are $\{(0, 1), (4, 1), (4, 0)\}$, and there are 3 elements of order 2. The elements of order 4 are $\{(2, 0), (2, 1), (6, 0), (6, 1)\}$, and there are 4 element of order 4.
   - $\mathbb{Z}_4 \times \mathbb{Z}_4$:
     The elements have orders 1, 2, or 4. The elements of order 2 are $(2, 0)$, $(2, 2)$, and $(0, 2)$. Thus, there is 1 element of order 1 (identity), 3 elements of order 2, and the remainder have order 4, so there are 12 elements of order 4. These are all elements in $\mathbb{Z}_4 \times \mathbb{Z}_4$ which have an element of order 4 (namely 1 or 3) in either the first coordinate or the second.
   - $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$:
     Again, the element have order 1, 2, or 4. The elements of order 4 are those with an element of order 4 in the first component (the other components are $\mathbb{Z}_2$ which have only orders of 1 and 2). There are 2 elements of order 2 in $\mathbb{Z}_4$ (namely, 1 and 3) and there are $2 \times 4 = 8$ elements with 1 or 3 in the first component. Since there is only 1 element of order 1, there are $16 - 8 - 1 = 7$ elements of order 2.

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2. Prove that every abelian group of order 45 has an element of order 15. Does every abelian group of order 45 have an element of order 9?

   ► **Solution.** Let G be a group of order $45 = 3^2 \cdot 5$. By Cauchy's theorem for abelian groups, there is an element $a \in G$ of order 3 and an element b of order 5. Let $c = ab$. Then, since the group G is abelian, $c^n = a^n b^n$ for all integers n. In particular, $c^{15} = a^{15} b^{15} = (a^3)^5 (b^5)^3 = e^5 e^3 = e$. Therefore, the order of c divides 15, so it must be 1, 3, 5, or 15. Since $c^3 = a^3 b^3 = e b^3 = b^3 \neq e$ (because $|b| = 5$ and hence $b^3 \neq e$) and $c^5 = a^5 b^5 = a^2 e = a^2 \neq e$ (since $|a| = 3$), it follows that the order of c cannot be 3 or 5 (or 1), so the order must be 15. Thus, any group of order 45 must have an element of order 15.

   Let $G = \mathbb{Z}_5 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. The order of any element (r, s, t) is the least common multiple of $\{|r|, |s|, |t|\}$. Since $|r|$ is 1 or 5, $|s|$ and $|t|$ are 1 or 3, the possible orders of elements in G are 1, 3, 5, and 15. Thus, there are no elements of order 9 in G. ◄

3. The symmetry group of a nonsquare rectangle is an abelian group of order 4. Is it isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$?

   ▶ **Solution.** The nonidentity elements are the reflections across the two lines joining the midpoints of opposite sides, and the rotation by 180°. Each of these has order 2, so the group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. ◀

4. The set $G = \{1, 9, 16, 22, 29, 53, 74, 79, 81\}$ is a group under multiplication modulo 91. Determine the isomorphism class of this group.

   ▶ **Solution.** The group is an abelian group of order 9, so it is isomorphic to $\mathbb{Z}_9$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$. $\langle 9\rangle = \{1, 9, 81\}$ since $9^3 = 729 = 1 \pmod{91}$, and $\langle 16\rangle = \{1, 16, 16^2 = 74 \pmod{91}\}$ since $16^3 = 4096 = 1 \pmod{91}$. Since G has two distinct subgroups of order 3, it cannot be cyclic (cyclic groups have a unique subgroup of each order dividing the order of the group). Thus, G must be isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$. As an internal direct product,

$$G \cong \langle 9\rangle \times \langle 16\rangle.$$

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5. Let $G = \{1, 7, 17, 23, 49, 55, 65, 71\}$ under multiplication modulo 96. Express G as an external and an internal direct product of cyclic groups.

   ▶ **Solution.** First note that $\langle 7\rangle = \{1, 7, 49, 55\}$ and $\langle 17\rangle = \{1, 17\}$. Thus, G has two distinct subgroups of order 2, namely $\langle 49\rangle$ and $\rangle 17\rangle$, so the group is not cyclic. Hence it must be $\mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Since there is an element of order 4, it must be isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$. An internal direct product is

$$G \cong \langle 7\rangle \times \langle 17\rangle.$$

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6. The set $G = \{1, 4, 11, 14, 16, 19, 26, 29, 31, 34, 41, 44\}$ is a group under multiplication modulo 45. Write G as an external and an internal direct product of cyclic groups of prime-power order.

   ▶ **Solution.** $\langle 16\rangle = \{1, 16, 31\}$, $\langle 19\rangle = \{1, 19\}$, and $\langle 26\rangle = \{1, 26\}$. Since

$$\begin{aligned}
\langle 26\rangle \cap \langle 16\rangle\langle 19\rangle &= \{1, 26\} \cap \{1, 4, 16, 19, 31, 34\} = \{1\} \\
\langle 19\rangle \cap \langle 16\rangle\langle 26\rangle &= \{1, 19\} \cap \{1, 11, 16, 26, 31, 41\} = \{1\} \\
\langle 16\rangle \cap \langle 19\rangle\langle 26\rangle &= \{1, 16, 31\} \cap \{1, 19, 26, 44\} = \{1\}
\end{aligned}$$

it follows that

$$G = \langle 19\rangle\langle 26\rangle\langle 16\rangle = \langle 19\rangle \times \langle 26\rangle \times \langle 16\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3.$$

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