1. Calculate the number of elements of order 2 in each of the abelian groups $\mathbb{Z}_{16}$, $\mathbb{Z}_8 \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_4$, and $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Do the same for elements of order 4.

**Solution.**

- **$\mathbb{Z}_{16}$:**
  A cyclic group has a unique subgroup of order dividing the order of the group. Thus, $\mathbb{Z}_{16}$ has one subgroup of order 2, namely $\langle 8 \rangle$, which gives the only element of order 2, namely 8. There is one subgroup of order 4, namely $\langle 4 \rangle$, and this subgroup has 2 generators, each of order 4. Thus the 2 elements of order 4 in $\mathbb{Z}_{16}$ are 4 and 12.

- **$\mathbb{Z}_8 \times \mathbb{Z}_2$:**
  The order of $(r, s)$ is the least common multiple of the order of $r$ and $s$. Thus elements of order 2 in $\mathbb{Z}_8 \times \mathbb{Z}_2$ are $\{(0, 1), (4, 1), (4, 0)\}$, and there are 3 elements of order 2. The elements of order 4 are $\{(2, 0), (2, 1), (6, 0), (6, 1)\}$, and there are 4 elements of order 4.

- **$\mathbb{Z}_4 \times \mathbb{Z}_4$:**
  The elements have orders 1, 2, or 4. The elements of order 2 are $(2, 0), (2, 2),$ and $(0, 2)$. Thus, there is 1 element of order 1 (identity), 3 elements of order 2, and the remainder have order 4, so there are 12 elements of order 4. These are all elements in $\mathbb{Z}_4 \times \mathbb{Z}_4$ which have an element of order 4 (namely 1 or 3) in either the first coordinate or the second.

- **$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$:**
  Again, the element have order 1, 2, or 4. The elements of order 4 are those with an element of order 4 in the first component (the other components are $\mathbb{Z}_2$ which have only orders of 1 and 2). There are 2 elements of order 2 in $\mathbb{Z}_4$ (namely, 1 and 3) and there are $2 \times 4 = 8$ elements with 1 or 3 in the first component. Since there is only 1 element of order 1, there are $16 - 8 - 1 = 7$ elements of order 2.

2. Prove that every abelian group of order 45 has an element of order 15. Does every abelian group of order 45 have an element of order 9?

**Solution.** Let $G$ be a group of order $45 = 3^2 \cdot 5$. By Cauchy’s theorem for abelian groups, there is an element $a \in G$ of order 3 and an element $b$ of order 5. Let $c = ab$. Then, since the group $G$ is abelian, $c^n = a^n b^n$ for all integers $n$. In particular, $c^{15} = a^{15} b^{15} = (a^3)^5 (b^5)^3 = e^5 e^3 = e$. Therefore, the order of $c$ divides 15, so it must be 1, 3, 5, or 15. Since $c^3 = a^3 b^3 = c b^3 = b^3 \neq e$ (because $|b| = 5$ and hence $b^3 \neq e$) and $c^5 = a^5 b^5 = a^2 e = a^2 \neq e$ (since $|a| = 3$), it follows that the order of $c$ cannot be 3 or 5 (or 1), so the order must be 15. Thus, any group of order 45 must have an element of order 15.

Let $G = \mathbb{Z}_5 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. The order of any element $(r, s, t)$ is the least common multiple of $\{|r|, |s|, |t|\}$. Since $|r|$ is 1 or 5, $|s|$ and $|t|$ are 1 or 3, the possible orders of elements in $G$ are 1, 3, 5, and 15. Thus, there are no elements of order 9 in $G$. 

Math 4200
3. The symmetry group of a nonsquare rectangle is an abelian group of order 4. Is it isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$?

▶ Solution. The nonidentity elements are the reflections across the two lines joining the midpoints of opposite sides, and the rotation by $180^\circ$. Each of these has order 2, so the group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. ◀

4. The set $G = \{1, 9, 16, 22, 29, 53, 74, 79, 81\}$ is a group under multiplication modulo 91. Determine the isomorphism class of this group.

▶ Solution. The group is an abelian group of order 9, so it is isomorphic to $\mathbb{Z}_9$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$. $\langle 9 \rangle = \{1, 9, 81\}$ since $9^3 = 729 = 1 \pmod{91}$, and $\langle 16 \rangle = \{1, 16, 16^2 = 74 \pmod{91}\}$ since $16^3 = 4096 = 1 \pmod{91}$. Since $G$ has two distinct subgroups of order 3, it cannot be cyclic (cyclic groups have a unique subgroup of each order dividing the order of the group). Thus, $G$ must be isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$. As an internal direct product,

\[ G \cong \langle 9 \rangle \times \langle 16 \rangle. \]

▶

5. Let $G = \{1, 7, 17, 23, 49, 55, 65, 71\}$ under multiplication modulo 96. Express $G$ as an external and an internal direct product of cyclic groups.

▶ Solution. First note that $\langle 7 \rangle = \{1, 7, 49\}$ and $\langle 17 \rangle = \{1, 17\}$. Thus, $G$ has two distinct subgroups of order 2, namely $\langle 49 \rangle$ and $\langle 17 \rangle$, so the group is not cyclic. Hence it must be $\mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Since there is an element of order 4, it must be isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$. An internal direct product is

\[ G \cong \langle 7 \rangle \times \langle 17 \rangle. \]

▶

6. The set $G = \{1, 4, 11, 14, 16, 19, 26, 29, 31, 34, 41, 44\}$ is a group under multiplication modulo 45. Write $G$ as an external and an internal direct product of cyclic groups of prime-power order.

▶ Solution. $\langle 16 \rangle = \{1, 16, 31\}$, $\langle 19 \rangle = \{1, 19\}$, and $\langle 26 \rangle = \{1, 26\}$. Since

\[
\langle 26 \rangle \cap \langle 16 \rangle = \{1, 26\} \cap \{1, 4, 16, 19, 31, 34\} = \{1\}
\]

\[
\langle 19 \rangle \cap \langle 16 \rangle = \{1, 19\} \cap \{1, 11, 16, 26, 31, 41\} = \{1\}
\]

\[
\langle 16 \rangle \cap \langle 19 \rangle = \{1, 16, 31\} \cap \{1, 19, 26, 44\} = \{1\}
\]

it follows that

\[ G = \langle 19 \rangle \langle 26 \rangle \langle 16 \rangle = \langle 19 \rangle \times \langle 26 \rangle \times \langle 16 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3. \]

▶