Do the following exercises from Judson:
Section 16.6: 8,13 (b)

1. If F is a field and $|F| = q$, show that $a^q = a$ for all $a \in F$.
2. Show that $\mathbb{Z}[\sqrt{2}] = \left\{m + n\sqrt{2} | m,\, n \in \mathbb{Z}\right\}$ is a subring of $\mathbb{C}$ and find 10 units.
3. In each case decide whether A is an ideal of the ring R.
   (a) $R = \mathbb{Z} \times \mathbb{Z}$, $A = \{(k,\, k) : k \in \mathbb{Z}\}$
   (b) $R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a,\, b,\, c \in \mathbb{Z} \right\}$, $A = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a,\, c \in \mathbb{Z},\, b \in 2\mathbb{Z} \right\}$.
4. Let $R = \mathbb{Z}[i]$ be the ring of gaussian integers and let $A = R(1 + 3i) = \langle 1 + 3i \rangle$. Find the number of elements in the factor ring R/A and describe the cosets.
5. Find all maximal ideals of (a) $\mathbb{Z}_8$, (b) $\mathbb{Z}_{10}$, (c) $\mathbb{Z}_{12}$, (d) $\mathbb{Z}_n$.
6. (a) Show that $\mathbb{Z}_3[\sqrt{2}]$ is a field.
   (b) Show that $\mathbb{Z}_2[\sqrt{2}]$ has a unique proper ideal $A \neq 0$.
7. Show that $\mathbb{Z} \times 0$ and $0 \times \mathbb{Z}$ are prime ideals of $\mathbb{Z} \times \mathbb{Z}$. Are they maximal ideals?
8. The nonzero elements of $\mathbb{Z}_3[i]$ form an abelian group of order 8 (since $\mathbb{Z}_3[i]$ is a field). Determine the isomorphism class of this group.