Instructions. Answer each of the questions on your own paper and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.

1. [15 Points] Determine (with justification) if each of the following real numbers is constructible.
   (a) \(\sqrt{2}\)  
   (b) \(\sqrt{10} + 2\sqrt{5}\)

2. [15 Points] Complete the following statements of theorems.
   (a) Let \(F\) be a finite field with \(p^n\) elements. Then \(F\) is a splitting field of the polynomial over the prime subfield \(\mathbb{Z}_p\) of \(F\).
   (b) If \(F\) is a finite field, then the multiplicative group \(F^*\) is a group.
   (c) Let \(K\) be a field, \(f(x) \in K[x]\) a polynomial of positive degree, and \(F\) a splitting field for \(f(x)\) over \(K\). If no irreducible factor of \(f(x)\) has repeated roots, then \(|\text{Gal}(F/K)|\) = \(\ldots\)

3. [20 Points] Let \(f(x) = x^4 - 9 \in \mathbb{Q}[x]\).
   (a) Find the factorization of \(f(x)\) as a product of irreducible polynomials in \(\mathbb{Q}[x]\).
   (b) Find all the roots of \(f(x)\) in the complex numbers \(\mathbb{C}\).
   (c) Find the splitting field \(K\) of \(f(x)\) over \(\mathbb{Q}\).
   (d) Find \([K : \mathbb{Q}]\). Justify your answer.

4. [20 Points] Determine all of the subfields of the Galois field \(GF(4096)\) and give the inclusion relations among these subfields. Note that \(4096 = 2^{12}\). Is \(GF(256)\) one of the subfields of \(GF(4096)\)?

5. [30 Points] Let \(F = \mathbb{Q}(\sqrt{3}, \sqrt{5})\) and let \(G = \text{Gal}(F/\mathbb{Q})\).
   (a) Find a polynomial \(f(x) \in \mathbb{Q}[x]\) so that \(F\) is a splitting field for \(f(x)\) over \(\mathbb{Q}\).
   (b) Explain why \(|G| = 4\). (That is, quote the appropriate theorem.)
   (c) If \(\sigma \in G\), explain why \(\sigma(\sqrt{3}) \in \{\sqrt{3}, -\sqrt{3}\}\) and \(\sigma(\sqrt{5}) \in \{\sqrt{5}, -\sqrt{5}\}\).
   (d) Since \(|G| = 4\) we can write \(G = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}\) where \(\sigma_0\) is the identity. Complete the following table giving the effect of each element of \(G\) on the linear basis \(\{1, \sqrt{3}, \sqrt{5}, \sqrt{15}\}\) of \(F\) over \(\mathbb{Q}\).
   
<table>
<thead>
<tr>
<th>1</th>
<th>(\sqrt{3})</th>
<th>(\sqrt{5})</th>
<th>(\sqrt{15})</th>
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<tbody>
<tr>
<td>(\sigma_0)</td>
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<tr>
<td>(\sigma_1)</td>
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<td>(\sigma_2)</td>
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<td>(\sigma_3)</td>
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   (e) Since \(G\) is a group of order 4, it must be isomorphic to either \(\mathbb{Z}_4\) or \(\mathbb{Z}_2 \times \mathbb{Z}_2\) (the Klein 4-group). Which group is \(G\) isomorphic to? Justify your answer.