3. In this exercise we outline how to construct a regular pentagon. Let \( \zeta = \cos(2\pi/5) + i\sin(2\pi/5) \).

(a) Show that \( \zeta \) is a primitive fifth root of unity.

**Solution.** From DeMoivre’s Theorem (see Page 446) \( \zeta^j = \cos(2j\pi/5) + i\sin(2j\pi/5) \neq 1 \) for \( j = 1, 2, 3, 4 \), but \( \zeta^5 = \cos(2\pi) + i\sin(2\pi) = 1 \).

(b) Show that \( (\zeta + \zeta^{-1})^2 + (\zeta + \zeta^{-1}) - 1 = 0 \).

**Solution.** First note that \( \zeta \) is a root of the polynomial \( x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1) \) and since \( \zeta \neq 1 \) it follows that \( \zeta^4 + \zeta^3 + \zeta^2 + \zeta + 1 = 0 \). Since \( \zeta^5 = 1 \) it follows that \( \zeta^{-j} = \zeta^{5-j} \). Then

\[
(\zeta + \zeta^{-1})^2 + (\zeta + \zeta^{-1}) - 1 = \zeta^2 + 2 + \zeta^{-2} + \zeta + \zeta^{-1} - 1 \\
= \zeta^2 + \zeta^1 + \zeta + \zeta^{-1} + 1 \\
= \zeta^2 + \zeta^3 + \zeta + \zeta^4 + 1 \\
= \zeta^4 + \zeta^3 + \zeta^2 + \zeta + 1 = 0.
\]

(c) Show that \( \zeta + \zeta^{-1} = (-1 + \sqrt{5})/2 \).

**Solution.** Since \( \zeta + \zeta^{-1} \) is a solution of the quadratic equation \( x^2 + x - 1 = 0 \), it follows that \( \zeta + \zeta^{-1} \) is either

\[
\frac{-1 + \sqrt{1 - 4(-1)}}{2} \quad \text{or} \quad \frac{-1 - \sqrt{1 - 4(-1)}}{2}.
\]

Since

\[
\zeta + \zeta^{-1} = (\cos(2\pi/5) + i\sin(2\pi/5)) + (\cos(2\pi/5) - i\sin(2\pi/5)) = 2\cos(2\pi/5) > 0,
\]

it follows that we must have \( \zeta + \zeta^{-1} = (-1 + \sqrt{5})/2 \).

(d) Show that \( \cos(2\pi/5) = (-1 + \sqrt{5})/4 \) and that \( \sin(2\pi/5) = \left( \sqrt{10 + 2\sqrt{5}} \right)/4 \).

**Solution.** From part (c), \( \cos(2\pi/5) = \frac{1}{2}(\zeta + \zeta^{-1}) = (-1 + \sqrt{5})/4 \). Since \( \sin(2pi/5) \) is in the first quadrant, it follows that \( \sin(2\pi/5) > 0 \). Thus,

\[
\sin(2\pi/5) = \sqrt{1 - \cos^2(2\pi/5)} = \sqrt{1 - \left( \frac{-1 + \sqrt{5}}{4} \right)^2} \\
= \sqrt{1 - \frac{1 - 2\sqrt{5} + 5}{16}} = \frac{1}{4} \sqrt{10 + 2\sqrt{5}}.
\]
Conclude that a regular pentagon is constructible.

- **Solution.** The construction of a regular \( n \)-gon is equivalent to constructing an angle of size \( 2\pi/n \), since such an angle can be duplicated \( n \)-times about a common vertex and a unit circle drawn centered at the common vertex. Thus, to construct a regular \( n \)-gon it is only necessary to construct the coordinates \( \cos(2\pi/n) \) and \( \sin(2\pi/n) \) on the unit circle centered at the origin. To see that \( \cos(2\pi/5) \) and \( \sin(2\pi/5) \) are constructible, note that both are in the field \( \mathbb{Q}(\sqrt{10} + 2\sqrt{5}) \) and apply Theorem 6.3.6 with \( u_1 = \sqrt{5} \) and \( u_2 = \sqrt{10} + 2\sqrt{5} \). Then \( u_1^2 = 5 \in \mathbb{Q} \) and \( u_2^2 = 10 + 2\sqrt{5} \in \mathbb{Q}(\sqrt{5}) = \mathbb{Q}(u_1) \). Thus, the theorem implies that any elements of \( \mathbb{Q}(\sqrt{10} + 2\sqrt{5}) \) are constructible, including \( \cos(2\pi/5) \) and \( \sin(2\pi/5) \).

1. Determine the splitting fields in \( \mathbb{C} \) for the following polynomials (over \( \mathbb{Q} \)).

- (b) \( x^2 + 3 \)

  - **Solution.** The roots are \( \pm \sqrt{3}i \) so the splitting field over \( \mathbb{Q} \) is \( \mathbb{Q}(\sqrt{3}i) \), since this field contains both of the roots and is the smallest such field.

- (d) \( x^3 - 5 \)

  - **Solution.** The polynomial factors as \( x^3 - 5 = (x - \sqrt[3]{5})(x - \omega \sqrt[3]{5})(x - \omega^2 \sqrt[3]{5}) \) where \( \omega = (-1 + \sqrt{3}i)/2 \) is a cube root of 1. Since \( \omega \notin \mathbb{Q}(\sqrt[3]{5}) \), the splitting field of \( x^3 - 5 \) over \( \mathbb{Q} \) is \( \mathbb{Q}(\sqrt[3]{5}, \omega) \).

2. Determine the splitting fields in \( \mathbb{C} \) for the following polynomials (over \( \mathbb{Q} \)).

- (a) \( x^3 - 1 \)

  - **Solution.** We have \( x^3 - 1 = (x - 1)(x^2 + x + 1) = (x - 1)(x - \omega)(x - \omega^2) \), where \( \omega = (-1 + \sqrt{3}i)/2 \). Thus \( x^3 - 1 \) splits in \( \mathbb{Q}(\omega) \) and hence, \( \mathbb{Q}(\omega) \) is the splitting field for \( x^3 - 1 \) over \( \mathbb{Q} \) since \( \omega \notin \mathbb{Q} \).

- (b) \( x^4 - 1 \)

  - **Solution.** We have \( x^4 - 1 = (x^2 + 1)(x^2 - 1) = (x + i)(x - i)(x + 1)(x - 1) \). Hence \( x^4 - 1 \) splits over \( \mathbb{Q}(i) \). Since \( i \notin \mathbb{Q} \), it follows that \( \mathbb{Q}(i) \) is the splitting field for \( x^4 - 1 \) over \( \mathbb{Q} \).

4. Let \( p \) be a prime number. Determine the splitting field in \( \mathbb{C} \) for \( x^p - 1 \) (over \( \mathbb{Q} \)).
Solution. Let $\zeta = e^{2\pi/p}$. Then $(\zeta^j)^p = e^{2j\pi} = 1$ and $\zeta^j \neq \zeta^k$ if $0 \leq j < k \leq p$. Thus $\zeta^j$ for $0 \leq j \leq p$ are $p$ distinct roots of the polynomial $x^p - 1$. Hence the polynomial splits as

$$x^p - 1 = (x - 1)(x - \zeta)(x - \zeta^2) \cdots (x - \zeta^{p-1}).$$

Thus, all the roots of $x^p - 1$ are in the field $\mathbb{Q}(\zeta)$, so $\mathbb{Q}(\zeta)$ is the splitting field of $x^p - 1$. ▶

Supplemental Exercises (i.e., not from the text).

1. Show that each of the following real numbers is constructible by applying Theorem 6.3.6. That is, for each number find the explicit sequence of real numbers $u_1, \ldots, u_n$ referred to in the theorem.

   (a) $\gamma_1 = 3\sqrt{2} + \sqrt{5 - 3\sqrt{2}}$

   Solution. Let $u_1 = \sqrt{2}$ and $u_2 = \sqrt{5 - 3\sqrt{2}}$. Then $u_1^2 = 2 \in \mathbb{Q}$, $u_2^2 = 5 - 3\sqrt{2} \in \mathbb{Q}(u_1)$ and $\gamma_1 \in \mathbb{Q}(u_1, u_2)$. Thus $\gamma_1$ is constructible. ▶

   (b) $\gamma_2 = 5\sqrt{2} + \frac{\sqrt{8 - 3\sqrt{2}}}{1 - \sqrt{2}}$

   Solution. Let $u_1 = \sqrt{2}$ and $u_2 = \sqrt{8 - 3\sqrt{2}}$. Then $u_1^2 = 2 \in \mathbb{Q}$, $u_2^2 = 8 - 3\sqrt{2} \in \mathbb{Q}(u_1)$ and $\gamma_2 \in \mathbb{Q}(u_1, u_2)$. Thus $\gamma_2$ is constructible. ▶

2. Show that the real roots of the polynomial $2x^4 - 6x^2 - 3$ are constructible. Hint: This polynomial is a quadratic in $x^2$.

Solution. Solve for $x^2$ by the quadratic formula:

$$x^2 = \frac{6 \pm \sqrt{6^2 - 24}}{4} = \frac{3 \pm \sqrt{15}}{2}.$$

The real roots of the original polynomial are then

$$\alpha_{1,2} = \pm \frac{\sqrt{3 + \sqrt{15}}}{\sqrt{2}} = \pm \frac{\sqrt{6 + 2\sqrt{15}}}{2}. $$

To show that these are constructible use Theorem 6.3.6. Let $u_1 = \sqrt{15}$ and $u_2 = \sqrt{6 + 2\sqrt{15}}$. Then $u_1^2 = 15 \in \mathbb{Q}$, $u_2^2 = 6 + 2\sqrt{15} \in \mathbb{Q}(u_1)$ and $\alpha_{1,2} \in \mathbb{Q}(u_1, u_2)$. Thus, the real roots $\alpha_{1,2}$ are constructible. ▶