Instructions. Answer each of the questions on your own paper and put your name on each page of your paper.

1. Give an example of each of the following. No proofs are required *for this exercise only.*

   (a) A nonabelian group of order 12.

   ▶ **Solution.** Two possible examples are the dihedral group D_{12} of order 12, and the alternating group A_4. ◀

   (b) An abelian, but noncyclic group of order 12.

   ▶ **Solution.** $\mathbb{Z}_2 \times \mathbb{Z}_6$. Several of you gave $\mathbb{Z}_3 \times \mathbb{Z}_4$ as an example. But this group is in fact cyclic of order 12. (See Exercise 11 (c).) ◀

   (c) A normal subgroup H of the symmetric group S_3, and a non-normal subgroup K of the symmetric group S_3.

   ▶ **Solution.** $H = \langle (1\ 2\ 3) \rangle$ and $K = \langle (1\ 2) \rangle$. ◀

   (d) An element σ of order 12 in the alternating group A_{10}.

   ▶ **Solution.** $\sigma = (1\ 2\ 3\ 4)(5\ 6\ 7\ 8\ 9\ 10)$ or $\sigma = (1\ 2\ 3\ 4)(5\ 6\ 7)(8\ 9)$ ◀

   (e) An element b of order 5 in the cyclic group $G = \langle a \rangle$ of order 20.

   ▶ **Solution.** $b = a^4$. ◀

2. Let G be the group of invertible 2×2 upper triangular matrices with entries in $\mathbb{R}$. That is,

$$G = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a,\ b,\ d \in \mathbb{R}, \text{ with } ad \neq 0 \right\}.$$

   Let

$$D = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} : ad \neq 0 \right\} \subseteq G$$

   be the subgroup of invertible diagonal matrices and let $U \subseteq G$ be the subgroup of matrices of the form $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ where $b \in \mathbb{R}$ is arbitrary. For this exercise, you may assume without proof that G, D, and U are groups under the operation of matrix multiplication.

   (a) Define a function $\varphi : G \to G$ by $\varphi\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$. Show that φ is a group homomorphism.

▶ **Solution.** $\varphi\left(\begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix}\begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix}\right) = \varphi\left(\begin{bmatrix} a_1a_2 & a_1b_2 + b_1d_2 \\ 0 & d_1d_2 \end{bmatrix}\right) = \begin{bmatrix} a_1a_2 & 0 \\ 0 & d_1d_2 \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ 0 & d_1 \end{bmatrix}\begin{bmatrix} a_2 & 0 \\ 0 & d_2 \end{bmatrix} = \varphi\left(\begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix}\right)\varphi\left(\begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix}\right)$. Thus φ is a group homomorphism. ◀

(b) Show that U is a normal subgroup of G and that G/U is isomorphic to D.

▶ **Solution.** Since $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in \operatorname{Ker}(\varphi)$ if and only if $a = d = 1$, it follows that $U = \operatorname{Ker}(\varphi)$. Since kernels of homomorphisms are normal subgroups, it follows that U is a normal subgroup of G. Moreover, $D = \operatorname{Im}(\varphi)$, so by the first isomorphism theorem,

$$G/U = G/\operatorname{Ker}(\varphi) \cong \operatorname{Im}(\varphi) = D.$$

◀

(c) **True** or **False** (with justification): G is the internal direct product of U and D.

▶ **Solution.** The criterion for G to be the internal direct product of U and D (Definition 6.3) requires that *both* U and D are normal subgroups of G. But $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \in D$, and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in G$, but

$$BAB^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \notin D,$$

so D is not normal in G. Therefore, G is **not** the internal direct product of U and D. ◀

3. Let G be a group of order $2006 = 2 \times 17 \times 59$. Can a group of this order be simple? Prove that your answer is correct. (Recall that a group G is simple if the only *normal* subgroups of G are $\{e\}$ and G.)

▶ **Solution.** Let $n_{59}(G)$ be the number of 59-Sylow subgroups of G. According the Sylow's theorem, $n_{59}(G) \equiv 1 \pmod{59}$ and $n_{59}(G) | 34 = 2006/59$. Thus, $n_{59}(G) = 1$ so there is exactly one subgroup of G of order 59. Let H be this subgroup of order 59. Since conjugating a subgroup of order 59 produces another subgroup of order 59, it follows that $gHg^{-1} = H$ for all $g \in G$, so H is normal in G. Thus G is not simple. ◀

4. (a) Let G be a group and $a \in G$. State (without proof) the formula that relates the size of the conjugacy class $[a]_C$ of a and the size of the centralizer $C(a)$ of a.

▶ **Solution.** $|[a]_C| = [G : C(a)] = |G| / |C(a)|$, assuming that $|G| < \infty$ (Page 15). ◀

(b) If $\sigma = (1\ 2)(3\ 4) \in S_4$, compute the number of elements of S_4 that are conjugate to σ and the number of elements of S_4 that commute with σ.

▶ **Solution.** The permutations that are conjugate to σ are all of those with the same cyclic type. Since σ is a product of two disjoint 2-cycles in S_4, there are

$$\frac{1}{2}\left(\frac{4 \cdot 3}{2} \times \frac{2 \cdot 1}{2}\right) = 3$$

elements in the conjugacy class $[\sigma]_C$. The elements of S_4 that commute with σ are the members of the centralizer $C(\sigma)$, and by the formula in part (a),

$$|C(\sigma)| = \frac{|S_4|}{|[\sigma]_C|} = \frac{24}{3} = 8.$$

It was not part of the question, but it is easy to list the 8 permutations that commute with σ: (1), (1 2)(3 4), (1 2), (3 4), (1 3)(2 4), (1 4)(2 3), (1 3 2 4), (1 4 2 3). To see where these come from, note that the first, second, third, and fourth elements form a subgroup of S_4 of order 4, and hence all elements commute. The same is true of the first, second, fifth and sixth elements, and the first, second, seventh and eighth elements. ◀