1. Let G be the abelian group with generators x, y, and z subject to the relations

$$\begin{aligned} 5x + 9y + 5z &= 0 \\ 2x + 4y + 2z &= 0 \\ x + \ \ y - 3z &= 0. \end{aligned}$$

   Determine the elementary divisors of G and write G as a direct sum of cyclic groups.

2. (a) Let R be a UFD (Unique Factorization Domain) and let $I \neq 0$ be a non-zero *prime* ideal. Show that there is a principal prime ideal $\langle a \rangle$ of R such that $\langle 0 \rangle \neq \langle a \rangle \subseteq I$. (*Hint:* Factor a non-zero element of I.)

   (b) Suppose that J is an ideal of $\mathbb{Z}$ with $J \neq \mathbb{Z}$, and suppose that J is not prime. Show that J does not contain a non-zero principal prime ideal.

3. Assume that the minimal polynomial $m_A(X)$ and the characteristic polynomial $c_A(X)$ are as given in the three cases below. Describe or write down all possible Jordan canonical forms for A.

   (a) $c_A(X) = (X-3)^4(X-2)^3$ and $m_A(X) = (X-3)^3(X-2)$.

   (b) $c_A(X) = (X-2)^5$ and $m_A(X) = (X-2)^3$.

   (c) $c_A(X) = (X-\lambda_1)^{e_1}(X-\lambda_2)^{e_2}\cdots(X-\lambda_m)^{e_m}$ and
   $m_A(X) = (X-\lambda_1)^{e_1-1}(X-\lambda_2)^{e_2-1}\cdots(X-\lambda_m)^{e_m-1}$
   for integers $e_1 \geq 2, \ldots, e_m \geq 2$.

4. Let A and $B \in M_4(\mathbb{Q})$ be the following matrices:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & k \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

   There is exactly one value of $k \in \mathbb{Q}$ for which A and B are similar. What is it?

5. (a) Let R be a ring and M an R-module. What does it mean for M to be a *free* R-module?

   (b) Let p be a prime and let $\mathbb{Z}\left[\frac{1}{p}\right]$ denote the subring of $\mathbb{Q}$ generated by $\mathbb{Z}$ and $\frac{1}{p}$. Give an explicit description of all rational numbers $\frac{a}{b}$ which are in $\mathbb{Z}\left[\frac{1}{p}\right]$ and prove that $\mathbb{Z}\left[\frac{1}{p}\right]$ is not a free $\mathbb{Z}$-module.

6. Let V be a finite dimensional vector space over a field $\mathbb{F}$ and let U and W be subspaces. Prove that if $\dim U + \dim W > \dim V$, then $U \cap W \neq \{0\}$.