1. Let R be an integral domain and let M be an R-module. Give the definition of each of the following terms:

   (a) M is a *free* R-module.

   (b) M is a *cyclic* R-module.

   Now determine if each of the following statements about R-modules is true or false. Give a proof or counterexample, as appropriate.

   (a) A submodule of a free module is free.

   (b) A submodule of a cyclic module is cyclic.

   (c) A quotient module of a free module is free.

   (d) A quotient module of a cyclic module is cyclic.

2. List without repetition all of the abelian groups of order $72 = 3^2 2^3$. Identify which group on your list is isomorphic to each of the following groups.

   (a) $\mathbb{Z}_{72}$

   (b) $\mathbb{Z}_4 \times \mathbb{Z}_{18}$

   (c) $\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_6$

3. Let $K \subseteq \mathbb{Z}^3$ be the $\mathbb{Z}$-submodule generated by $v_1 = (1, 0, -1)$ and $v_2 = (5, 6, 7)$.

   (a) Find a basis $\{x_1,\ x_2,\ x_3\}$ of $\mathbb{Z}^3$ and integers s_1 and s_2, with $s_1 | s_2$, so that $\{s_1 x_1,\ s_2 x_2\}$ is a basis of K.

   (b) What is the order of each of the elements $x_i + K$ in the quotient group $\mathbb{Z}^3/K$?

   (c) Determine the structure of $\mathbb{Z}^3/K$ as a direct sum of cyclic submodules.

4. Let V be a vector space over the real numbers $\mathbb{R}$ and let $T : V \to V$ be a linear transformation. Describe how V can be made into an $\mathbb{R}[X]$-module (denoted V_T) via T.

   Suppose that V has a basis $\mathcal{B} = \{e_1,\ e_2,\ e_3\}$ and T is the linear transformation defined by $T(e_1) = 2e_1$, $T(e_2) = -4e_2 - 4e_3$, and $T(e_3) = 9e_2 + 8e_3$.

   (a) Compute the matrix representation $[T]_{\mathcal{B}}$.

   (b) Calculate $(X - 2)e_1$ and $(X - 2)^2 e_2$.

   (c) Show that $V_T = \mathbb{R}[X]e_1 \oplus \mathbb{R}[X]e_2$.

   (d) What are the invariant factors of T?

   (e) Find the Jordan canonical form J of T and a basis $\mathcal{C}$ for V such that $[T]_{\mathcal{C}} = J$.

5. Let $A \in M_n(\mathbb{R})$ be a real matrix such that $A^2 = I_n$. Describe all possible Jordan canonical forms for the matrix A.