Math 7200
Final Examination
December 13, 1995

(8) 1. Determine (with proof) whether there is a $\mathbf{Z}$-module homomorphism $\phi : \mathbf{Z}^3 \to \mathbf{Z}^3$ with $\phi(1,0,1) = (1,1,-1)$, $\phi(0,1,-1) = (2,-1,1)$, and $\phi(2,1,1) = (4,1,1)$.

(12) 2. (a) Give an example (with justification) of a short exact sequence of finite rank $\mathbf{Z}$-modules

$$(*) \qquad 0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

with M not isomorphic to $N \oplus K$.

(b) If we request, instead, that the short exact sequence $(*)$ be a sequence of finite rank $\mathbf{C}$-modules, explain (using the appropriate theorems) why $M \cong N \oplus K$.

(20) 3. Let $K \subseteq \mathbf{Z}^3$ be the $\mathbf{Z}$-submodule generated by $v_1 = (1,0,-1)$ and $v_2 = (5,6,7)$.

(a) Find a basis $\{x_1, x_2, x_3\}$ of $\mathbf{Z}^3$ and integers s_1 and s_2, with $s_1 \mid s_2$, so that $\{s_1 x_1, s_2 x_2\}$ is a basis of K.

(b) Determine the structure of $\mathbf{Z}^3/K$.

(c) What is $\mu(\mathbf{Z}^3/K)$?

(20) 4. Let $V = \mathbf{Q}^2$ and define $T \in \operatorname{End}_{\mathbf{Q}}(V)$ by

$$T(u_1, u_2) = (2u_1 - u_2, u_1 + 4u_2).$$

Let $\mathcal{B} = \{e_1, e_2\}$ be the standard basis on V.

(a) Compute the matrix $A = [T]_{\mathcal{B}}$.

(b) Let V_T be the vector space V made into a $\mathbf{Q}[X]$-module by T. Compute $X \cdot e_1$, $(X-3)\cdot e_1$, and $(X-3)^2 \cdot e_1$.

(c) Determine $\operatorname{Ann}(V_T)$.

(d) Show that e_1 is a generator for the $\mathbf{Q}[X]$-module V_T.

(e) If $\mathcal{B}' = \{e_1, X \cdot e_1\}$, compute the matrix $[T]_{\mathcal{B}'}$.

(20) 5. Let

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 \\ -1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 5 \end{bmatrix} \in M_4(\mathbf{Q}).$$

Find the Jordan form J of A and an invertible matrix P such that $P^{-1}AP = J$.

(20) 6. Let F be a field and let $T \in \operatorname{End}_F(V)$ where V is a vector space over F. Assume the following facts about T:

$$c_T(X) = (X-2)^6 (X-4)^4 (X+3)^2$$
$$m_T(X) = (X-2)^3 (X-4)^3 (X+3)$$
$$\nu_{\text{geom}}(2) = 3.$$

Compute the Jordan canonical form of T, $\mu(V_T)$, the invariant factors of T, and the geometric multiplicities of the eigenvalues 4 and -3.