**Math 7200**
**Make-up Midterm Examination**
**October 19, 1999**

1. Let $G = \langle a \rangle$ be a cyclic group of order 24.
   (a) Let $b = a^6$, $c = a^4$, $H = \langle b \rangle$, and $K = \langle c \rangle$. What is $|H|$ and $|K|$? Prove that your answers are correct.
   (b) Determine generators for each of the subgroups $H \cap K$ and $HK$, and determine the orders of each of these subgroups.
   (c) Is $G$ the internal direct product of $H$ and $K$? Justify your answer.
2. Let $V$ be a finite dimensional vector space over a field $F$ and let $W$ be a subspace. The *codimension* of $W$, denoted $\operatorname{codim} W$, is defined by the formula

$$\operatorname{codim} W = \dim V - \dim W.$$

   If $W_1$ and $W_2$ are subspaces of $V$, prove that

$$\operatorname{codim}(W_1 \cap W_2) \leq \operatorname{codim} W_1 + \operatorname{codim} W_2.$$

3. Let $V$ be a vector space of dimension 3 over $\mathbf{C}$, let $\mathcal{B} = (v_1,\, v_2,\, v_3)$ be an ordered basis for $V$ and let $T \in L(V)$ be the linear operator defined by

$$T(v_1) = 0, \quad T(v_2) = -v_1, \quad T(v_3) = 5v_1 + v_2.$$

   (a) Compute the matrix $[T]_{\mathcal{B}}$.
   (b) Show that $T$ is nilpotent, i.e., $T^k = 0$ for some $k$.
   (c) Determine a basis $\mathcal{C} = (w_1,\, w_2,\, w_3)$ of $V$ so that the matrix $[T]_{\mathcal{C}}$ is in Jordan canonical form.
4. Let $T$ be a linear operator on the complex vector space $V$. Assume that you are given the following data concerning $T$:
   (a) $c_T(X) = (X-1)^4(X-2)^5(X-3)$
   (b) $\nu_{\text{geom.}}(1) = 2$ and $\nu_{\text{geom.}}(2) = 1$. Find all Jordan forms for $T$ consistent with these data, and for each possible Jordan form, give the minimal polynomial.