Math 7200
Final Examination
December 11, 1999

Directions: Do any two of the three problems in each of Parts I and II. Please start each problem on a new sheet of paper, with the problem number written at the top of every sheet. Hand in only the four problems you wish to have graded. Good luck!

Part I.

1. Let
$$F = \left\{ \begin{bmatrix} a & b \\ 5b & a \end{bmatrix} : a, b \in \mathbb{Q} \right\}.$$
   (a) Prove that F is a field under the usual matrix operations of addition and multiplication.
   (b) Prove that F is isomorphic to the field $\mathbb{Q}(\sqrt{5})$.
2. Let $p \in \mathbb{Z}$ be a prime and let $I_p = \langle X^3 + X + 1,\, p\rangle \subseteq \mathbb{Z}[X]$ be the ideal of $\mathbb{Z}[X]$ generated by p and $f(X) = X^3 + X + 1$.
   (a) If $p = 2$, prove that I_2 is a maximal ideal of $\mathbb{Z}[X]$.
   (b) If $p = 3$, prove that I_3 is not a maximal ideal and find a maximal ideal M of $\mathbb{Z}[X]$ which contains I_3.
3. Let R be a commutative ring with identity and let I and J be ideals of R.
   (a) Define
$$(I : J) = \{r \in R : rx \in I \text{ for all } x \in J\}.$$
   Show that $(I : J)$ is an ideal of R containing I.
   (b) Show that if P is a prime ideal of R and $x \notin P$, then $(P : \langle x\rangle) = P$, where $\langle x\rangle$ denotes the principal ideal generated by x.
   (c) If $R = \mathbb{Z}$, $I = \langle 200\rangle$, and $J = \langle 55\rangle$, determine the ideal $(I : J)$.

Part II.

4. Let R be a ring and let I be an ideal of R. Prove that I is a free R-module if and only if I is a principal ideal generated by an element α which is not a zero divisor.
5. List all abelian groups of order 8 up to isomorphism. Identify which group on your list is isomorphic to each of the following groups of order 8. Justify your answer.
   (a) $(\mathbb{Z}/15\mathbb{Z})^* =$ the group of units of the ring $\mathbb{Z}/15\mathbb{Z}$.
   (b) The roots of the equation $z^8 - 1 = 0$ in $\mathbb{C}$.
   (c) $\mathbb{F}_8^+ =$ the additive group of the field $\mathbb{F}_8$ with eight elements.
6. Let G be an abelian group generated by x, y, z subject to the relations
$$\begin{aligned} 15x + 3y &= 0 \\ 3x + 7y + 4z &= 0 \\ 18x + 14y + 8z &= 0. \end{aligned}$$
   (a) Write G as a product of two cyclic groups.
   (b) Write G as a direct product of cyclic groups of prime power order.
   (c) How many elements of G have order 2?