1. (a) Show that the matrix $A \in M_3(\mathbb{F})$ ($\mathbb{F}$ a field) is uniquely determined up to similarity by the characteristic polynomial $c_A(X)$ and the minimal polynomial $m_A(X)$.

   (b) Give an example of two matrices A, $B \in M_4(\mathbb{F})$ with the same characteristic and minimal polynomials, but with A and B not similar.

2. In each case below, you are given some of the following information for a linear transformation $T : V \to V$, V a vector space over the complex numbers $\mathbb{C}$: (1) characteristic polynomial for T; (2) minimal polynomial for T; (3) algebraic multiplicity of each eigenvalue; (4) geometric multiplicity of each eigenvalue; (5) the elementary divisors of the module V_T. Find all possibilities for T consistent with the given data (up to similarity) and for each possibility give the rational and Jordan canonical forms and the rest of the data. *Since there are a number of cases needed for each part, pick any two of the following situations (a) – (f), and work them out fully.*

   (a) $c_T(X) = (X-2)^4(X-3)^2$.

   (b) $c_T(X) = X^2(X-4)^7$ and $m_T(X) = X(X-4)^3$.

   (c) $\dim V = 6$ and $m_T(X) = (X+3)^2(X+1)^2$.

   (d) $c_T(X) = X(X-1)^4(X-2)^5$, $\nu_{\mathrm{geom}}(1) = 2$, and $\nu_{\mathrm{geom}}(2) = 2$.

   (e) $c_T(X) = (X-5)(X-7)(X-9)(X-11)$.

   (f) $\dim V = 4$ and $m_T(X) = X - 1$.

3. Find the characteristic polynomial, minimal polynomial, and Jordan canonical form of the linear transformation $T : \mathbb{C}^3 \to \mathbb{C}^3$ with matrix

$$\begin{bmatrix} 4 & 0 & 4 \\ 2 & 1 & 3 \\ -1 & 0 & 0 \end{bmatrix}.$$

4. Show that the matrices

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

   are similar in $M_3(\mathbb{Z}_3)$, but are not similar in $M_5(\mathbb{Z}_5)$.