1. (a) Show that the matrix $A \in M_3(\mathbb{F})$ ($\mathbb{F}$ a field) is uniquely determined up to similarity by the characteristic polynomial $c_A(X)$ and the minimal polynomial $m_A(X)$.

▶ **Solution.** A matrix $A \in M_3(\mathbb{F})$ determines a linear transformation $T_A : \mathbb{F}^3 \to \mathbb{F}^3$ by matrix multiplication $T_A(v) = Av$ (viewing elements of $V = \mathbb{F}^3$ as 3×1 column matrices). Then the relevant result is that A is similar to $B \iff T_A$ is similar to $T_B \iff V_{T_A}$ is isomorphic to V_{T_B} as $F[X]$ modules $\iff$ the invariant factors of V_{T_A} are equal to the invariant factors of V_{T_B}. Thus what we need to show is that we can recover the invariant factors of V_{T_A} from a knowledge of $m_A(X) = m_{T_A}(X)$ and $c_A(X) = c_{T_A}(X)$. But $m_A(X)$ is the invariant factor of largest degree and $c_A(X)$ is the product of *all* invariant factors. We will proceed by considering the possible degrees of $m_A(X)$.

Case 1. $\deg m_A(X) = 1$. In this case, $m_A(X) = X - r$ for some $r \in \mathbb{F}$, and since every irreducible polynomial that divides $c_A(X)$ must also divide $m_A(X)$, it follows that $c_A(X) = (X - r)^3$ (since $\deg c_A(X) = 3$). Thus, in this case, the invariant factors of V_{T_A} must be $f_1(X) = f_2(X) = f_3(X) = X - r$.

Case 2. $\deg m_A(X) = 2$ Since $m_A(X)$ divides $c_A(X)$ and $\deg c_A(X) = 3$, we must have $c_A(X) = (X - t)m_A(X)$ for some $t \in \mathbb{F}$. Since every prime divisor of $c_A(X)$ must also divide $m_A(X)$ it follows that $(X - t)$ divides $m_A(X)$, and hence the only possibility for the invariant factors is $f_1(X) = X - t$ and $f_2(X) = m_A(X)$. Hence, in this case $m_A(X)$ and $c_A(X)$ determine the invariant factors.

Case 3. $\deg m_A(X) = 3$. In this case, $m_A(X) = c_A(X)$ and there is only one invariant factor, namely $f_1(X) = m_A(X)$, which again is known once $m_A(X)$ is known. ◀

(b) Give an example of two matrices $A, B \in M_4(\mathbb{F})$ with the same characteristic and minimal polynomials, but with A and B not similar.

▶ **Solution.** $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ have $m_A(X) = m_B(X) = X^2$ and $c_A(X) = C_B(X) = X^4$. But they are not similar since both are in Jordan canonical form and these forms are different. (The geometric multiplicity of the eigenvalue 0 is 3 for A and 2 for B.) ◀

2. In each case below, you are given some of the following information for a linear transformation $T : V \to V$, V a vector space over the complex numbers $\mathbb{C}$: (1) characteristic polynomial for T; (2) minimal polynomial for T; (3) algebraic multiplicity of each eigenvalue; (4) geometric multiplicity of each eigenvalue; (5) the elementary divisors of the module V_T. Find all possibilities for T consistent with the given data (up to similarity) and for each possibility give the rational and Jordan canonical forms and the rest of the data. *Since there are a number of cases needed for each part, pick any two of the following situations (a) – (f), and work them out fully.*

(a) $c_T(X) = (X - 2)^4(X - 3)^2$.

▶ **Solution.** The similarity class of T is determined by the elementary divisors, which will be a list of prime powers $(X-r)^k$ for $r=2$, or $r=3$. The product of the powers $(X-2)^{k_i}$ must be $(X-2)^4$ and the product of the powers $(X-3)^{\ell_j}$ must be $(X-3)^2$ since the product of all the elementary divisors is $c_T(X)$. Thus, we need to list all the ways to write 4 as a sum of positive integers and 2 as a sum of positive integers. Hence, each similarity class of T is determined by the pair consisting of a partition of 4 and a partition of 2. These are:

Partitions of 4	Partitions of 2
4	2
3+1	1+1
2+2	
2+1+1	
1+1+1+1	

Thus there are 10 similarity classes of linear transformations T with $c_T(S) = (X-2)^4(X-3)^2$. For each of these classes, we will list the requested data using the following notation: $J_{\lambda,n}$ is the Jordan block with size n and eigenvalue λ, $C(f(X))$ is the companion matrix of the polynomial $f(X)$, $\mathrm{diag}(d_1, \ldots, d_n)$ is the $n \times n$ diagonal matrix with diagonal entries $d_1, \ldots, d_n$. A 1×1 matrix with entry r will be denoted $[r]$. We will organize our data by listing the ordered pair (Partition of 4, Partition of 2) followed by the information about T.

•

Partition Pair:	(4, 2)
Elementary Divisors:	$(X-2)^4$, $(X-3)^2$
Invariant Factors:	$(X-2)^4(X-3)^2$
$m_T(X)$	$(X-2)^4(X-3)^2$
Jordan Canonical Form:	$J_{2,4} \oplus J_{3,2}$
Rational Canonical Form:	$C((X-2)^4(X-3)2)$
ν_{alg}	$\nu_{\mathrm{alg}}(2) = 4$, $\nu_{\mathrm{alg}}(3) = 2$
ν_{geom}	$\nu_{\mathrm{geom}}(2) = 1$, $\nu_{\mathrm{geom}}(3) = 1$

•

Partition Pair:	$(4, 1+1)$
Elementary Divisors:	$(X-2)^4$, $X-3$, $X-3$
Invariant Factors:	$(X-2)^4(X-3)$, $X-3$
$m_T(X)$	$(X-2)^4(X-3)$
Jordan Canonical Form:	$J_{2,4} \oplus \mathrm{diag}(3, 3)$
Rational Canonical Form:	$C((X-2)^4(X-3)) \oplus [3]$
ν_{alg}	$\nu_{\mathrm{alg}}(2) = 4$, $\nu_{\mathrm{alg}}(3) = 2$
ν_{geom}	$\nu_{\mathrm{geom}}(2) = 1$, $\nu_{\mathrm{geom}}(3) = 2$

Partition Pair:	$(3+1,\ 2)$
Elementary Divisors:	$(X-2)^3,\ X-2,\ (X-3)^2$
Invariant Factors:	$(X-2)^3(X-3)^2,\ X-2$
$m_T(X)$	$(X-2)^3(X-3)^2$
Jordan Canonical Form:	$J_{2,3} \oplus J_{2,1} \oplus J_{3,2}$
Rational Canonical Form:	$C((X-2)^3(X-3)^2) \oplus [2]$
$\nu_{\rm alg}$	$\nu_{\rm alg}(2) = 4,\ \nu_{\rm alg}(3) = 2$
$\nu_{\rm geom}$	$\nu_{\rm geom}(2) = 2,\ \nu_{\rm geom}(3) = 1$

Partition Pair:	$(3+1,\ 1+1)$
Elementary Divisors:	$(X-2)^3,\ X-2,\ X-3,\ X-3$
Invariant Factors:	$(X-2)^3(X-3),\ (X-2)(X-3)$
$m_T(X)$	$(X-2)^3(X-3)$
Jordan Canonical Form:	$J_{2,3} \oplus \mathrm{diag}(2,\ 3,\ 3)$
Rational Canonical Form:	$C((X-2)^3(X-3)) \oplus C((X-2)(X-3))$
$\nu_{\rm alg}$	$\nu_{\rm alg}(2) = 4,\ \nu_{\rm alg}(3) = 2$
$\nu_{\rm geom}$	$\nu_{\rm geom}(2) = 2,\ \nu_{\rm geom}(3) = 2$

Partition Pair:	$(2+2,\ 2)$
Elementary Divisors:	$(X-2)^2,\ (X-2)^2,\ (X-3)^2$
Invariant Factors:	$(X-2)^2(X-3)^2,\ (X-2)^2$
$m_T(X)$	$(X-2)^2(X-3)^2$
Jordan Canonical Form:	$J_{2,2} \oplus J_{2,2} \oplus J_{3,2}$
Rational Canonical Form:	$C((X-2)^2(X-3)^2) \oplus C((X-2)^2)$
$\nu_{\rm alg}$	$\nu_{\rm alg}(2) = 4,\ \nu_{\rm alg}(3) = 2$
$\nu_{\rm geom}$	$\nu_{\rm geom}(2) = 2,\ \nu_{\rm geom}(3) = 1$

Partition Pair:	$(2+2,\ 1+1)$
Elementary Divisors:	$(X-2)^2,\ (X-2)^2,\ X-3,\ X-3$
Invariant Factors:	$(X-2)^2(X-3),\ (X-2)^2(X-3)$
$m_T(X)$	$(X-2)^2(X-3)$
Jordan Canonical Form:	$J_{2,2} \oplus J_{2,2} \oplus \mathrm{diag}(3,\ 3)$
Rational Canonical Form:	$C((X-2)^2(X-3)) \oplus C((X-2)^2(X-3))$
$\nu_{\rm alg}$	$\nu_{\rm alg}(2) = 4,\ \nu_{\rm alg}(3) = 2$
$\nu_{\rm geom}$	$\nu_{\rm geom}(2) = 2,\ \nu_{\rm geom}(3) = 2$

-

Partition Pair:	$(2+1+1,\ 2)$
Elementary Divisors:	$(X-2)^2,\ X-2,\ X-2,\ (X-3)^2$
Invariant Factors:	$(X-2)^2(X-3)^2,\ X-2,\ X-2$
$m_T(X)$	$(X-2)^2(X-3)^2$
Jordan Canonical Form:	$J_{2,2} \oplus \text{diag}(2,\ 2) \oplus J_{3,2}$
Rational Canonical Form:	$C((X-2)^2(X-3)^2) \oplus \text{diag}(2,\ 2)$
ν_{alg}	$\nu_{\text{alg}}(2) = 4,\ \nu_{\text{alg}}(3) = 2$
ν_{geom}	$\nu_{\text{geom}}(2) = 3,\ \nu_{\text{geom}}(3) = 1$

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Partition Pair:	$(2+1+1,\ 1+1)$
Elementary Divisors:	$(X-2)^2,\ X-2,\ X-2,\ X-3,\ X-3$
Invariant Factors:	$(X-2)^2(X-3),\ (X-2)(X-3),\ X-2$
$m_T(X)$	$(X-2)^2(X-3)$
Jordan Canonical Form:	$J_{2,2} \oplus \text{diag}(2,\ 2,\ 3,\ 3)$
Rational Canonical Form:	$C((X-2)^2(X-3)) \oplus C((X-2)(X-3)) \oplus [2]$
ν_{alg}	$\nu_{\text{alg}}(2) = 4,\ \nu_{\text{alg}}(3) = 2$
ν_{geom}	$\nu_{\text{geom}}(2) = 3,\ \nu_{\text{geom}}(3) = 2$

-

Partition Pair:	$(1+1+1+1,\ 2)$
Elementary Divisors:	$X-2,\ X-2,\ X-2,\ X-2,\ (X-3)^2$
Invariant Factors:	$(X-2)(X-3)^2,\ X-2,\ X-2,\ X-2$
$m_T(X)$	$(X-2)(X-3)^2$
Jordan Canonical Form:	$\text{diag}(2,\ 2,\ 2,\ 2) \oplus J_{3,2}$
Rational Canonical Form:	$C((X-2)(X-3)^2) \oplus \text{diag}(2,\ 2,\ 2)$
ν_{alg}	$\nu_{\text{alg}}(2) = 4,\ \nu_{\text{alg}}(3) = 2$
ν_{geom}	$\nu_{\text{geom}}(2) = 4,\ \nu_{\text{geom}}(3) = 1$

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Partition Pair:	$(1+1+1+1,\ 1+1)$
Elementary Divisors:	$X-2,\ X-2,\ X-2,\ X-2,\ X-3,\ X-3$
Invariant Factors:	$(X-2)(X-3),\ (X-2)(X-3),\ X-2,\ X-2$
$m_T(X)$	$(X-2)(X-3)$
Jordan Canonical Form:	$\text{diag}(2,\ 2,\ 2,\ 2,\ 3,\ 3)$
Rational Canonical Form:	$C((X-2)(X-3)) \oplus C((X-2)(X-3)) \oplus \text{diag}(2,\ 2)$
ν_{alg}	$\nu_{\text{alg}}(2) = 4,\ \nu_{\text{alg}}(3) = 2$
ν_{geom}	$\nu_{\text{geom}}(2) = 4,\ \nu_{\text{geom}}(3) = 2$

◀

(b) $c_T(X) = X^2(X-4)^7$ and $m_T(X) = X(X-4)^3$.

▶ **Solution.** In this case we need to look for the partitions of 2 with maximum summand of 1 and partitions of 7 with a maximum summand of 3, so that the highest degree invariant factor (that is $m_T(X)$) has the factorization $m_T(X) = X(X-4)^3$. Thus, there is only one valid partition of 2, namely $1+1$, and 4 valid partitions of 7: $3+3+1$, $3+2+2$, $3+2+1+1$, $3+1+1+1+1$. As in the above exercise, the results will be presented in tabular form.

Partition Pair:	$(3+3+1,\ 1+1)$
Elementary Divisors:	$(X-4)^3,\ (X-4)^3,\ X-4,\ X,\ X$
Invariant Factors:	$X(X-4)^3,\ X(X-4)^3,\ X-4$
Jordan Canonical Form:	$J_{4,3} \oplus J_{4,3} \oplus \mathrm{diag}(4,0,0)$
Rational Canonical Form:	$C(X(X-4)^3) \oplus C(X(X-4)^3) \oplus [0]$
ν_{alg}	$\nu_{\mathrm{alg}}(0) = 2,\ \nu_{\mathrm{alg}}(4) = 7$
ν_{geom}	$\nu_{\mathrm{geom}}(0) = 2,\ \nu_{\mathrm{geom}}(4) = 3$

Partition Pair:	$(3+2+2,\ 1+1)$
Elementary Divisors:	$(X-4)^3,\ (X-4)^2,\ (X-4)^2,\ X,\ X$
Invariant Factors:	$X(X-4)^3,\ X(X-4)^2,\ (X-4)^2$
Jordan Canonical Form:	$J_{4,3} \oplus J_{4,2} \oplus J_{4,2} \oplus \mathrm{diag}(0,0)$
Rational Canonical Form:	$C(X(X-4)^3) \oplus C(X(X-4)^2) \oplus C((X-4)^2)$
ν_{alg}	$\nu_{\mathrm{alg}}(0) = 2,\ \nu_{\mathrm{alg}}(4) = 7$
ν_{geom}	$\nu_{\mathrm{geom}}(0) = 2,\ \nu_{\mathrm{geom}}(4) = 3$

Partition Pair:	$(3+2+1+1,\ 1+1)$
Elementary Divisors:	$(X-4)^3,\ (X-4)^2,\ X-4,\ X-4,\ X,\ X$
Invariant Factors:	$X(X-4)^3,\ X(X-4)^2,\ X-4,\ X-4$
Jordan Canonical Form:	$J_{4,3} \oplus J_{4,2} \oplus \mathrm{diag}(4,\ 4,\ 0,\ 0)$
Rational Canonical Form:	$C(X(X-4)^3) \oplus C(X(X-4)^2) \oplus \mathrm{diag}(4,\ 4)$
ν_{alg}	$\nu_{\mathrm{alg}}(0) = 2,\ \nu_{\mathrm{alg}}(4) = 7$
ν_{geom}	$\nu_{\mathrm{geom}}(0) = 2,\ \nu_{\mathrm{geom}}(4) = 4$

Partition Pair:	$(3+1+1+1+1,\ 1+1)$
Elementary Divisors:	$(X-4)^3,\ X-4,\ X-4,\ X-4,\ X-4,\ X,\ X$
Invariant Factors:	$X(X-4)^3,\ X(X-4),\ X-4,\ X-4,\ X-4$
Jordan Canonical Form:	$J_{4,3} \oplus \mathrm{diag}(4,\ 4,\ 4,\ 4,\ 0,\ 0)$
Rational Canonical Form:	$C(X(X-4)^3) \oplus C(X(X-4)) \oplus \mathrm{diag}(4,\ 4,\ 4)$
ν_{alg}	$\nu_{\mathrm{alg}}(0) = 2,\ \nu_{\mathrm{alg}}(4) = 7$
ν_{geom}	$\nu_{\mathrm{geom}}(0) = 2,\ \nu_{\mathrm{geom}}(4) = 5$

◀

(c) $\dim V = 6$ and $m_T(X) = (X+3)^2(X+1)^2$.

▶ **Solution.** Since $m_T(X)$ divides $c_T(X)$ and they have the same irreducible factors, it follows that we must have

$$c_T(X) = (X+3)^i(X+1)^j$$

where $i + j = 6 = \dim V$ and $i \geq 2$, $j \geq 2$. Because the highest power of $X+3$ and $X+1$ appearing in $m_T(X)$ is 2, it follows that any Jordan block has maximal size 2.

-

(i, j):	(2, 4)
Partition Pair:	$(2,\ 2+2)$
Elementary Divisors:	$(X+3)^2$, $(X+1)^2$, $(X+1)^2$
Invariant Factors:	$(X+3)^2(X+1)^2$, $(X+1)^2$
$c_T(X)$	$(X+3)^2(X+1)^4$
Jordan Canonical Form:	$J_{-3,2} \oplus J_{-1,2} \oplus J_{-1,2}$
Rational Canonical Form:	$C((X+3)^2(X+1)^2) \oplus C((X+1)^2)$
ν_{alg}	$\nu_{\text{alg}}(-3) = 2$, $\nu_{\text{alg}}(-1) = 4$
ν_{geom}	$\nu_{\text{geom}}(-3) = 1$, $\nu_{\text{geom}}(-1) = 2$

-

(i, j):	(2, 4)
Partition Pair:	$(2,\ 2+1+1)$
Elementary Divisors:	$(X+3)^2$, $(X+1)^2$, $X+1$, $X+1$
Invariant Factors:	$(X+3)^2(X+1)^2$, $X+1$, $X+1$
$c_T(X)$	$(X+3)^2(X+1)^4$
Jordan Canonical Form:	$J_{-3,2} \oplus J_{-1,2} \oplus \operatorname{diag}(-1,\ -1)$
Rational Canonical Form:	$C((X+3)^2(X+1)^2) \oplus \operatorname{diag}(-1,\ -1)$
ν_{alg}	$\nu_{\text{alg}}(-3) = 2$, $\nu_{\text{alg}}(-1) = 4$
ν_{geom}	$\nu_{\text{geom}}(-3) = 1$, $\nu_{\text{geom}}(-1) = 3$

-

(i, j):	(4, 2)
Partition Pair:	$(2+2,\ 2)$
Elementary Divisors:	$(X+3)^2$, $(X+3)^2$, $(X+1)^2$
Invariant Factors:	$(X+3)^2(X+1)^2$, $(X+3)^2$
$c_T(X)$	$(X+3)^4(X+1)^2$
Jordan Canonical Form:	$J_{-3,2} \oplus J_{-3,2} \oplus J_{-1,2}$
Rational Canonical Form:	$C((X+3)^2(X+1)^2) \oplus C((X+3)^2)$
ν_{alg}	$\nu_{\text{alg}}(-3) = 4$, $\nu_{\text{alg}}(-1) = 2$
ν_{geom}	$\nu_{\text{geom}}(-3) = 2$, $\nu_{\text{geom}}(-1) = 1$

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(i, j):	(4, 2)
Partition Pair:	$(2+1+1,\ 2)$
Elementary Divisors:	$(X+3)^2,\ X+3,\ X+3,\ (X+1)^2$
Invariant Factors:	$(X+3)^2(X+1)^2,\ X+3,\ X+3$
$c_T(X)$	$(X+3)^4(X+1)^2$
Jordan Canonical Form:	$J_{-3,\,2} \oplus \mathrm{diag}(-3,\ -3) \oplus J_{-1,\,2}$
Rational Canonical Form:	$C((X+3)^2(X+1)^2) \oplus \mathrm{diag}(-3,\ -3)$
ν_{alg}	$\nu_{\mathrm{alg}}(-3) = 4,\ \nu_{\mathrm{alg}}(-1) = 2$
ν_{geom}	$\nu_{\mathrm{geom}}(-3) = 3,\ \nu_{\mathrm{geom}}(-1) = 1$

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(i, j):	(3, 3)
Partition Pair:	(3, 3)
Elementary Divisors:	$(X+3)^2,\ X+3,\ (X+1)^2,\ X+1$
Invariant Factors:	$(X+3)^2(X+1)^2,\ (X+1)(X+3)$
$c_T(X)$	$(X+3)^3(X+1)^3$
Jordan Canonical Form:	$J_{-3,\,2} \oplus J_{-1,\,2} \oplus \mathrm{diag}(-1,\ -3)$
Rational Canonical Form:	$C((X+3)^2(X+1)^2) \oplus C((X+1)(X+3))$
ν_{alg}	$\nu_{\mathrm{alg}}(-3) = 3,\ \nu_{\mathrm{alg}}(-1) = 3$
ν_{geom}	$\nu_{\mathrm{geom}}(-3) = 2,\ \nu_{\mathrm{geom}}(-1) = 2$

◀

(d) $c_T(X) = X(X-1)^4(X-2)^5$, $\nu_{\mathrm{geom}}(1) = 2$, and $\nu_{\mathrm{geom}}(2) = 2$.

▶ **Solution.** The conditions on the geometric multiplicity mean that the Jordan Canonical Form of T must have two blocks with eigenvalue 1 and two blocks with eigenvalue 2. Thus the elementary divisors will be of the form X, $(X-1)^i$, $(X-1)^j$, $(X-2)^k$, $(X-2)^l$ where i, j, k, and l are not zero and $i+j=4$, $k+l=5$. Thus there are two choices for $(i,\ j)$: (2, 2) and (3, 1) and two choices for $(k,\ l)$: (4, 1) and (3, 2). We thus get the following 4 possibilities for T:

•

$(i,\ j)$ and $(k,\ l)$:	(2, 2), (4, 1)
Elementary Divisors:	$X,\ (X-1)^2,\ (X-1)^2,\ (X-2)^4,\ X-2$
Invariant Factors:	$X(X-1)^2(X-2)^4,\ (X-1)^2(X-2)$
$m_T(X)$	$X(X-1)^2(X-2)^4$
Jordan Canonical Form:	$J_{1,\,2} \oplus J_{1,\,2} \oplus J_{2,\,4} \oplus \mathrm{diag}(0,\ 2)$
Rational Canonical Form:	$C(X(X-1)^2(X-2)^4) \oplus C((X-1)^2(X-2))$
ν_{alg}	$\nu_{\mathrm{alg}}(0) = 1,\ \nu_{\mathrm{alg}}(1) = 4,\ \nu_{\mathrm{alg}}(2) = 5$
ν_{geom}	$\nu_{\mathrm{geom}}(0) = 1,\ \nu_{\mathrm{geom}}(1) = 2,\ \nu_{\mathrm{geom}}(2) = 2$

•

$(i,\, j)$ and $(k,\, l)$:	(2, 2), (3, 2)
Elementary Divisors:	$X,\ (X-1)^2,\ (X-1)^2,\ (X-2)^3,\ (X-2)^2$
Invariant Factors:	$X(X-1)^2(X-2)^3,\ (X-1)^2(X-2)^2$
$m_T(X)$	$X(X-1)^2(X-2)^3$
Jordan Canonical Form:	$J_{1,2} \oplus J_{1,2} \oplus J_{2,3} \oplus J_{2,2} \oplus [0]$
Rational Canonical Form:	$C(X(X-1)^2(X-2)^3) \oplus C((X-1)^2(X-2)^2)$
ν_{alg}	$\nu_{\text{alg}}(0) = 1,\ \nu_{\text{alg}}(1) = 4,\ \nu_{\text{alg}}(2) = 5$
ν_{geom}	$\nu_{\text{geom}}(0) = 1,\ \nu_{\text{geom}}(1) = 2,\ \nu_{\text{geom}}(2) = 2$

•

$(i,\, j)$ and $(k,\, l)$:	(3, 1), (4, 1)
Elementary Divisors:	$X,\ (X-1)^3,\ X-1,\ (X-2)^4,\ X-2$
Invariant Factors:	$X(X-1)^3(X-2)^4,\ (X-1)(X-2)$
$m_T(X)$	$X(X-1)^3(X-2)^4$
Jordan Canonical Form:	$J_{1,3} \oplus J_{2,4} \oplus \operatorname{diag}(0,\, 1,\, 2)$
Rational Canonical Form:	$C(X(X-1)^3(X-2)^4) \oplus C((X-1)(X-2))$
ν_{alg}	$\nu_{\text{alg}}(0) = 1,\ \nu_{\text{alg}}(1) = 4,\ \nu_{\text{alg}}(2) = 5$
ν_{geom}	$\nu_{\text{geom}}(0) = 1,\ \nu_{\text{geom}}(1) = 2,\ \nu_{\text{geom}}(2) = 2$

•

$(i,\, j)$ and $(k,\, l)$:	(3, 1), (3, 2)
Elementary Divisors:	$X,\ (X-1)^3,\ X-1,\ (X-2)^3,\ (X-2)^2$
Invariant Factors:	$X(X-1)^3(X-2)^3,\ (X-1)(X-2)^2$
$m_T(X)$	$X(X-1)^3(X-2)^3$
Jordan Canonical Form:	$J_{1,3} \oplus J_{2,3} \oplus J_{2,2} \oplus \operatorname{diag}(0,\, 1)$
Rational Canonical Form:	$C(X(X-1)^3(X-2)^3) \oplus C((X-1)(X-2)^2)$
ν_{alg}	$\nu_{\text{alg}}(0) = 1,\ \nu_{\text{alg}}(1) = 4,\ \nu_{\text{alg}}(2) = 5$
ν_{geom}	$\nu_{\text{geom}}(0) = 1,\ \nu_{\text{geom}}(1) = 2,\ \nu_{\text{geom}}(2) = 2$

◀

(e) $c_T(X) = (X-5)(X-7)(X-9)(X-11)$.

▶ **Solution.** In this case there are 4 distinct eigenvalues (namely 5, 7, 9, 11), each of algebraic and geometric multiplicity 1. The elementary divisors are $X-5$, $X-7$, $X-9$, and $X-11$. The Jordan canonical form is the diagonal matrix $\operatorname{diag}(5,\, 7,\, 9,\, 11)$, while the rational canonical form is the companion matrix $C(c_T(X))$. ◀

(f) $\dim V = 4$ and $m_T(X) = X - 1$.

▶ **Solution.** If $m_T(X) = X - 1$, then the only possibility for T is $T = I$ (i.e., $T(v) = v$ for all $v \in V$). Since $\dim V = 4$, this gives $c_T(X) = (X-1)^4$; $\nu_{\text{alg}}(1) = 4$; $\nu_{\text{geom}}(1) = 4$; the elementary divisors are $X-1$, $X-1$, $X-1$, $X-1$. Both the rational and Jordan canonical forms are the identity matrix I. ◀

3. Find the characteristic polynomial, minimal polynomial, and Jordan canonical form of the linear transformation $T : \mathbb{C}^3 \to \mathbb{C}^3$ with matrix

$$A = \begin{bmatrix} 4 & 0 & 4 \\ 2 & 1 & 3 \\ -1 & 0 & 0 \end{bmatrix}.$$

► **Solution.** The characteristic polynomial is

$$\begin{aligned} c_A(X) = \det(XI - A) &= \det \begin{bmatrix} X-4 & 0 & -4 \\ -2 & X-1 & -3 \\ 1 & 0 & X \end{bmatrix} \\ &= (X-1)\det \begin{bmatrix} X-4 & -4 \\ 1 & X \end{bmatrix} \\ &= (X-1)(X^2 - 4X + 4) \\ &= (X-1)(X-2)^2. \end{aligned}$$

The candidates for the minimal polynomial are $(X-1)(X-2)$ and $c_A(X)$ since every prime divisor of $c_A(X)$ must also divide $m_A(X)$. But

$$(A - I)(A - 2I) = \begin{bmatrix} 3 & 0 & 4 \\ 2 & 0 & 3 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 4 \\ 2 & -1 & 3 \\ -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 4 \\ 1 & 0 & 2 \\ -1 & 0 & -2 \end{bmatrix} \neq 0.$$

Thus $m_A(X) \neq (X-1)(X-2)$ so $m_A(X) = c_A(X)$. Since $\operatorname{Rank}(A - 2I) = 2$ it follows that $\nu_{\text{geom}}(2) = 1$. Thus there is one block with eigenvalue 2 in the Jordan canonical form of A and the Jordan form is

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

By finding an eigenvector v_1 with eigenvalue 1, an eigenvector v_2 with eigenvalue 2, and a vector v_3 such that $(A - 2I)v_3 = v_2$, we can produce an invertible matrix P such that $P^{-1}AP = J$. For example, we can take

$$P = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

To see that this works, simply check that $AP = PJ$. ◄

4. Show that the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

are similar in $M_3(\mathbb{Z}_3)$, but are not similar in $M_3(\mathbb{Z}_5)$.

▶ **Solution.** Calculate that $c_A(X) = X^3 - 1$ and $c_B(X) = (X-1)^3$. In $\mathbb{Z}_5[X]$ these polynomials are not equal, so that A and B cannot be similar in $M_3(\mathbb{Z}_5)$. However, in the field $\mathbb{Z}_3$, we have that $(a+b)^3 = a^3 + b^3$. Thus $X^3 - 1 = (X-1)^3 \in \mathbb{Z}_3[X]$ so that $c_A(X) = c_B(X)$. Since

$$(A - I)^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

we conclude that $m_A(X) \neq (X-1)^2$ and hence it must be $(X-1)^3 = m_B(X)$. Hence, over $\mathbb{Z}_3$, A and B has the same characteristic and minimal polynomials. Exercise 1 then shows that A and B are similar. ◀