From the text (pages 45 – 48): 12, 23, 24. Do the following additional exercises:

1. Recall that

$$\mathrm{GL}_2(\mathbb{R}) = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,\, b,\, c,\, d \in \mathbb{R},\, \det A = ad - bc \neq 0 \right\}$$

is the group (under matrix multiplication) of invertible 2×2 real matrices. The following are two subsets of $\mathrm{GL}_2(\mathbb{R})$:

$$G = \left\{ \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} : x > 0,\, y \in \mathbb{R} \right\} \text{ and } H = \left\{ \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} : x > 0 \right\}.$$

   (a) Show that G and H are subgroups of $\mathrm{GL}_2(\mathbb{R})$.

   (b) Is H a normal subgroup of G?

   (c) The element $\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}$ of G can be identified with the point $(x,\, y)$ in the right half-plane of the cartesian plane $\mathbb{R}^2$. Using this identification, draw the partition of the right half plane into left cosets of H. Do the same for the right cosets of H?

2. (a) If a is an element of a group G, there is always a group homomorphism $\varphi : \mathbb{Z} \to G$ such that $\varphi(1) = a$. Give the formula for φ.

   (b) When is there a homomorphism from $\mathbb{Z}_n \to G$ which sends $[1]$ to a?

   (c) What are the homomorphisms from $\mathbb{Z}_2$ to $\mathbb{Z}_6$?

   (d) What are the homomorphisms from $\mathbb{Z}_4$ to $\mathbb{Z}_8$?

3. Suppose that G is a group and g is an element of G with $g \neq e$.

   (a) Under what conditions on g is there a homomorphism $f : \mathbb{Z}_7 \to G$ with $f([1]) = g$?

   (b) Under what conditions on g is there a homomorphism $f : \mathbb{Z}_{15} \to G$ with $f([1]) = g$?

   (c) Under what conditions on G is there an injective homomorphism $f : \mathbb{Z}_{15} \to G$?

   (d) Under what conditions on G is there a surjective homomorphism $f : \mathbb{Z}_{15} \to G$?

4. If G is a group let $Z(G) = \{a \in G : ab = ba \text{ for all } b \in G\}$. $Z(G)$ is called the *center* of G.

   (a) Verify that $Z(G)$ is an abelian subgroup of G.

   (b) If $H \subseteq Z(G)$ is any subgroup of the center of G, prove that H is a normal subgroup of G.

   (c) If $G = \mathrm{GL}_2(\mathbb{R})$ show that

$$Z(G) = \left\{ aI_2 = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in \mathbb{R}^* \right\}.$$