P45 #30

1. Suppose that H is a normal in G and suppose that $[G : H] = n$. We see that the quotient group G/H has order n. In particular we must have $\overline{a}^n = e$ for any $\overline{a} \in G/H$ (i.e. $a^n \in H$).

2. Take a subgroup generated by a transposition in S_3.

P45 #31

Done in class.

P45 #32

Let G be a p-group (i.e. $|G| = p^n$).

1. We know that $|G| = |Z(G)| + \sum[G : C(a)]$ and it is easy to see that p divides $[G : C(a)]$ for any $a \notin Z(G)$. In particular p divides $|G| - \sum[G : C(a)]$ and therefore divides $|Z(G)|$.

2. Suppose that $|G| = p^n$ and choose $a \in Z(G)$ such that $a \neq e$. Since $< a >$ is a subgroup of $Z(G)$, we see that $o(a) = p^k$ for some $k \neq 0$. In particular the order of $x = a^{p^{k-1}}$ must be p; hence $< x >$ is a normal subgroup of G of order p. In this framework, $G/ < x >$ is a group of cardinal p^{n-1} and we can conclude by induction.

Exercise 1

(1) $\sigma = (1\,3\,7\,5)(2\,6\,10\,4\,8)$; $o(\sigma) = 20$; σ is odd.
(2) $\tau = (1\,3\,7)(2\,5\,4)(6\,9)(8\,10)$; $o(\tau) = 6$; τ is even.
(3) $\sigma^2 = (1\,7)(3\,5)(2\,10\,8\,6\,4)$; $o(\tau) = 10$; σ is even.
(4) $\sigma\tau = (1\,7\,3\,5\,8\,4\,6\,9\,10\,2)$; $o(\sigma\tau) = 10$; σ is odd.
(5) $\tau\sigma = (1\,7\,4\,10\,2\,9\,6\,8\,5\,3)$; $o(\tau\sigma) = 10$; σ is odd.
(6) $\sigma^2\tau = (1\,5\,2\,3)(4\,10\,6\,9)$; $o(\tau\sigma) = 4$; σ is even.
(7) $\sigma\tau\sigma^{-1} = (1\,8\,6)(2\,4)(3\,7\,5)(9\,10)$; $o(\tau\sigma) = 6$; σ is even.

Exercise 2

σ and τ are conjugate by $(3\,4\,5\,7)$.

Exercise 3

1. Consider a set S with n elements. It is well known that the number of ordered subset of S of r elements is

$$P^n_r = \frac{n!}{(n-r)!}$$

But a r-cycle may be written by r different ways and therefore the number of r-cycles is

$$\frac{n!}{r(n-r)!} = \frac{1}{r}[(n)(n-1)\cdots(n-r+1)]$$

2.

cycle structure	number
(a)	1
$(a\,b)$	10
$(a\,b\,c)$	20
$(a\,b\,c\,d)$	30
$(a\,b\,c\,d\,e)$	24
$(a\,b)(c\,d)$	15
$(a\,b\,c)(d\,e)$	20

Exercise 4

(1) How many permutations in S_5 commute with (1 2 3), and how many *even* permutations commute with (1 2 3)? [*Hint:* Proposition (2.27) may be useful.]

Solution. Letting $G = S_5$ and $\sigma = (1,\ 2,\ 3)$, we see that

$$\begin{aligned}[G : C_G(\sigma)] &= \text{the number of conjugates of } \sigma \\ &= \text{the number of permutations with cycle structure } (1\,2,\ 3) \\ &= 20 \quad \text{(from the table in Exercise 3(b))}.\end{aligned}$$

Since $[G : C_G(\sigma)] = |G| \,/\, |C_G(\sigma)| = 20$ we conclude that $|C_G(\sigma)| = 120/20 = 6$. Thus there are 6 permutations in S_5 that commute with $\sigma = (1,\ 2,\ 3)$. Since the powers of σ and any permutation that is disjoint with σ commutes with σ, we can list these six explicitly:

$$(1),\ (1,\ 2,\ 3),\ (1,\ 3,\ 2), (4,\ 5),\ (1,\ 2,\ 3)(4,\ 5),\ (1,\ 3, 2)(4,\ 5)$$

The first three of these are even and the last three are odd.

(2) Same question for (1 2)(3 4).

Solution. Letting $G = S_5$ and $\sigma = (1,\ 2)(3,\ 4)$, we see that

$$\begin{aligned}[G : C_G(\sigma)] &= \text{the number of conjugates of } \sigma \\ &= \text{the number of permutations with cycle structure } (1\,2)(3,\ 4) \\ &= 15 \quad \text{(from the table in Exercise 3(b))}.\end{aligned}$$

Since $[G : C_G(\sigma)] = |G| \,/\, |C_G(\sigma)| = 15$ we conclude that $|C_G(\sigma)| = 120/15 = 8$. Thus there are 8 permutations in S_5 that commute with $\sigma = (1,\, 2)(3,\, 4)$. If $H = A_5$, then $C_H(\sigma) = H \cap C_G(\sigma)$ so that $C_H(\sigma) = C_G(\sigma)$ or if $C_H(\sigma) \neq C_G(\sigma)$ then there is an odd permutation $\alpha \in C_G(\sigma)$ and the multiplication map $\beta \mapsto \alpha\beta$ pairs up the even and odd elements of $C_G(\sigma)$. Hence, in this case exactly half of the elements of $C_G(\sigma)$ are even and half are odd. Since $\alpha = (1,\, 2)$ is odd and is in $C_G(\sigma)$, it follows that there are 4 even permutations that commute with σ.