From the text (pages 45 – 48): 49, 50, 52.

1. Let G act on a set X. Assume that $y = gx$ where $g \in G$ and $x, y \in X$. Prove that the stabilizers $G(x)$ and $G(y)$ are conjugate subgroups of G.

   ► **Solution.** Since $G(z) = \{a \in G : az = z\}$ we have that

$$\begin{aligned} G(y) &= \{a \in G : ay = y\} \\ &= \{a \in G : a(gx) = gx\} \\ &= \{a \in G : (ag)x = gx\} \\ &= \left\{a \in G : g^{-1}((ag)x) = g^{-1}(gx)\right\} \\ &= \left\{a \in G : (g^{-1}ag)x = g^{-1}((ag)x) = g^{-1}(gx) = (g^{-1}g)x = ex = x\right\}. \end{aligned}$$

   Thus, $a \in G(y) \iff g^{-1}ag \in G(x)$. Hence, $g^{-1}G(y)g = G(x)$, so $G(x)$ and $G(y)$ are conjugate subgroups of G. ◄

2. Let G be a p-group with $|G| = p^n$. Show that any subgroup of G of order p^{n-1} must be normal in G.

   ► **Solution.** Since $|G| = p^n$, it follows that p is the only prime dividing $|G|$, and hence it is the smallest prime dividing $|G|$. According to Corollary 4.6, Page 24, any subgroup H of G of index p must be normal in G. Since $[G : H] = p \iff |H| = p^{n-1}$, it follows that any subgroup H of G of order p^{n-1} must be normal. ◄

3. Suppose that $n \geq 3$. Is S_n isomorphic to a direct product $A_n \times G$ where G is a group of order 2? Naturally, a proof of your claim is required.

   ► **Solution.** Suppose that $\varphi : A_n \times G \to S_n$, where $|G| = 2$, is a group isomorphism. Then $H = \varphi(e_{A_n} \times G)$ is a normal subgroup of S_n of order 2, since $e_{A_n} \times G$ is a normal subgroup of $A_n \times G$ of order $|G| = 2$ (see the discussion at the top of page 35). But a subgroup H of S_n of order 2 is just $H = \{(1), \sigma\}$ where σ is a permutation of order 2. Since the order of a permutation is the least common multiple of the orders of the disjoint cycles in the disjoint cycle decomposition, it follows that σ must be a product of k disjoint 2-cycles, for some $k \geq 1$. Any two products of k disjoint 2-cycles in S_n are conjugate (Corollary 5.10, Page 33). Since $n \geq 3$, if $\sigma = (i_1\ j_1) \cdots (i_k\ j_k)$ and $m \notin \{i_1, j_1\}$, then $\tau = (j_1\ m)\sigma(j_1\ m) = (i_1\ m) \cdots \neq \sigma$. Hence $H = \{(1), \sigma\}$ is not closed under conjugation so H is not normal in S_n. Since we have shown that there is no normal subgroup of S_n of order 2, it is impossible to have an isomorphism $S_n \cong A_n \times G$. ◄

4. List all 3-Sylow subgroups of A_4 and list all 3-Sylow subgroups of S_4.

▶ **Solution.** Since $|S_4| = 24 = 3 \times 8$ and $|A_4| = 12$, it follows that a 3-Sylow subgroup of either A_4 or S_4 consists of a subgroup of order 3, which must then be cyclic since 3 is prime. The elements of order 3 in S_4 consist of a 3-cycle, and all such elements are also in A_4. Thus there are 4 3-Sylow subgroups of A_4 and S_4:

$$\begin{aligned} H_1 &= \langle (1\ 2\ 3) \rangle = \{(1),\ (1\ 2\ 3),\ (1\ 3\ 2)\} \\ H_2 &= \langle (1\ 2\ 4) \rangle = \{(1),\ (1\ 2\ 4),\ (1\ 4\ 2)\} \\ H_3 &= \langle (1\ 3\ 4) \rangle = \{(1),\ (1\ 3\ 4),\ (1\ 4\ 3)\} \\ H_4 &= \langle (2\ 3\ 4) \rangle = \{(1),\ (2\ 3\ 4),\ (2\ 4\ 3)\} \end{aligned}$$

◀

5. List all 2-Sylow subgroups of S_4 and find elements of S_4 which conjugate one of these into each of the others.

▶ **Solution.** Since $|S_4| = 24 = 8 \times 3$, it follows that a 2-Sylow subgroup of S_4 is a subgroup of order 8. The number of such 2-Sylow subgroups must be an odd number that divides 3, so there are 1 or 3 subgroups of S_4 of order 8. We claim that there are in fact 3 subgroups of S_4 of order 8. For an easy way to describe these subgroups, start with a square in the plane, for example with vertices at $(1,\ 1)$, $(-1,\ 1)$, $(-1,\ -1)$, and $(1,\ -1)$. Label these vertices as 1, 2, 3, and 4 in the order given. The group H of symmetries of this square (rotations by 90°, 180°, 270°, 360°; reflections through the two diagonals and through the x and y-axes), identified by the effect on the vertices, gives a subgroup of S_4 of order 8 (which is of course isomorphic to the dihedral group D_8. With these identifications we get a subgroup of S_4

$$H_1 = \{(1),\ (1\ 2\ 3\ 4),\ (1\ 3)(2\ 4),\ (1\ 4\ 3\ 2),\ (1\ 3),\ (2\ 4),\ (1\ 2)(3\ 4),\ (1\ 4)(2\ 3)\}$$

of order 8. If we now conjugate H_1 by each of the transpositions $(2\ 3)$ and $(3\ 4)$, we find that $(2\ 3)H_1(2\ 3)$ contains the two elements of order 4:

$$\begin{aligned} (1\ 3\ 2\ 4) &= (2\ 3)(1\ 2\ 3\ 4)(2\ 3) \\ (1\ 4\ 2\ 3) &= (2\ 3)(1\ 4\ 3\ 2)(2\ 3), \end{aligned}$$

while $(3\ 4)H_1(3\ 4)$ contains the two elements of order 4:

$$\begin{aligned} (1\ 2\ 4\ 3) &= (3\ 4)(1\ 2\ 3\ 4)(3\ 4) \\ (1\ 3\ 4\ 2) &= (3\ 4)(1\ 4\ 3\ 2)(3\ 4). \end{aligned}$$

Thus, H_1, $H_2 = (2\ 3)H_1(2\ 3)$, and $H_3 = (3\ 4)H_1(3\ 4)$ are three different subgroups of S_4 of order 8 (since each of them contains 2 of the six 4-cycles of S_4). Thus H_1, H_2 and H_3 are the 3 distinct 2-Sylow subgroups of S_4, and each of them is already described explicitly as a conjugate of H_1. ◀

49. Show that any group of order 312 has a nontrivial normal subgroup.

▶ **Solution.** Let G be a group of order 312. Since $312 = 8 \times 3 \times 13$, there is a 13-Sylow subgroup P of G which has order 13. Moreover, the number of such 13-Sylow subgroups is congruent to 1 mod 13 and divides 24, so there is exactly 1 such subgroup. Since there is only one subgroup of order 13, it is normal because any conjugate of a group of order 13 must be another group of order 13. ◀

50. Show that any group of order 56 has a nontrivial normal subgroup.

▶ **Solution.** Let G be a group of order $56 = 8 \times 7$. By Sylow's theorem there will be either 1 or 8 7-Sylow subgroups of G (since the number of such subgroups must be congruent to 1 mod 7 and divide 8). If there is only one subgroup of order 7, then it must be normal since any conjugate of a subgroup of order 7 must be another subgroup of order 7. If there are 8 distinct subgroups of order 7, then since 7 is prime, any two of these subgroups of order 7 can intersect only in the identity. Thus these 8 subgroups account for 49 of the 56 elements of G; namely the $8 \times 6 = 48$ elements of order 7 (6 in each of the 8 subgroups) plus the identity. Then remaining 7 elements plus the identity must then constitute the 2-Sylow subgroup of G, which has order 8. Thus there is a unique 2-Sylow subgroup of G, which must then be normal, in case there are more than one 7-Sylow subgroups. Hence, G has either a normal 7-Sylow subgroup or a normal 2-Sylow subgroup. ◀

52. How many elements are there of order 7 in a simple group of order 168?

▶ **Solution.** Let G be a simple group of order $168 = 7 \times 24$. Since G is not simple, there must be more than one 7-Sylow subgroup. As in the previous exercise, the number of 7-Sylow subgroups must be 8 and these 8 subgroups account for $8 \times 6 = 48$ elements of order 7. ◀