From the text (pages 98 – 106): 21, 22

1. If $I = \langle 1+2i \rangle$ is the principal ideal generated by $1+2i$ in the ring of Gaussian integers $\mathbb{Z}[i]$, then show that $\mathbb{Z}[i]/I$ is a finite field, and find its order.

2. Express the polynomial $X^4 - 2X^2 - 3$ as a product of irreducible polynomials over each of the following fields: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_5$.

3. Let R be the quadratic integer ring $\mathbb{Z}[\sqrt{-5}]$. Define 3 ideals of R: $I_2 = \langle 2,\ 1+\sqrt{-5} \rangle$, $I_3 = \langle 3,\ 2+\sqrt{-5} \rangle$, and $I_3' = \langle 3,\ 2-\sqrt{-5} \rangle$.

   (a) Prove that each of the ideals I_2, I_3 and I_3' is a nonprincipal ideal of R.

   (b) Show that $I_2^2 = \langle 2 \rangle$, so that the product of nonprincipal ideals can be a principal ideal.

   (c) Similarly, prove that $I_2 I_3 = \langle 1 - \sqrt{-5} \rangle$ and $I_2 I_3' = \langle 1 + \sqrt{-5} \rangle$ are principal. Deduce that the principal ideal $\langle 6 \rangle$ is a product of 4 ideals:

$$\langle 6 \rangle = I_2^2 I_3 I_3'.$$