1. Recall from class that if V is a vector space over a field $\mathbb{F}$ and $T : V \to V$ is a linear transformation, then the vector space V is made into an $\mathbb{F}[X]$ module V_T by defining the scalar multiplication
$$f(X)v = f(T)(v).$$
(See Examples 12 and 13, Pages 111-112.) For this exercise, we will let $\mathbb{F} = \mathbb{R}$, $V = \mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ will be the $\mathbb{R}$-linear transformation defined by the formula
$$T(x_1,\, x_2,\, \ldots,\, x_n) = (x_n,\, x_1,\, x_2,\, \ldots,\, x_{n-1}).$$
Answer the following questions concerning the $\mathbb{R}[X]$ module V_T determined by this linear transformation T. Let $v = (x_1,\, x_2,\, \ldots,\, x_n)$.
   (a) Compute Xv.
   (b) Compute $(X^2 + 2)v$.
   (c) Compute $(X^{n-1} + X^{n-2} + \cdots + X + 1)v$.
   (d) If $e_1 = (1,\, 0,\, \ldots,\, 0)$ and $f(X) = a_0 + a_1X + \cdots + a_{n-1}X^{n-1}$ is an arbitrary polynomial of degree $\leq n - 1$, compute $f(X)e_1$.
   (e) Verify that $(X^n - 1)v = 0$ for all $v \in V$.
   (f) Show that no polynomial $f(X)$ of degree $< n$ has the property $f(X)v = 0$ for all $v \in V$.

2. Let $\mathbb{F} = \mathbb{R}$ and let $V = \mathbb{R}^2$.
   (a) If $T_1 : V \to V$ is counterclockwise rotation about the origin by $\pi/2$ radians, show that V and $\{0\}$ are the only $\mathbb{R}[X]$ submodules of V_{T_1}.
   (b) If $T_2 : V \to V$ is counterclockwise rotation about the origin by π radians, show that *every* subspace of V is an $\mathbb{R}[X]$ submodule of V_{T_2}.