1. Recall from class that if V is a vector space over a field $\mathbb{F}$ and $T : V \to V$ is a linear transformation, then the vector space V is made into an $\mathbb{F}[X]$ module V_T by defining the scalar multiplication
$$f(X)v = f(T)(v).$$
(See Examples 12 and 13, Pages 111-112.) For this exercise, we will let $\mathbb{F} = \mathbb{R}$, $V = \mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ will be the $\mathbb{R}$-linear transformation defined by the formula
$$T(x_1, x_2, \ldots, x_n) = (x_n, x_1, x_2, \ldots, x_{n-1}).$$
Answer the following questions concerning the $\mathbb{R}[X]$ module V_T determined by this linear transformation T. Let $v = (x_1, x_2, \ldots, x_n)$.

   (a) Compute Xv.

   ▶ **Solution.** $Xv = T(v) = (x_n, x_1, x_2, \ldots, x_{n-1})$. ◀

   (b) Compute $(X^2 + 2)v$.

   ▶ **Solution.**
$$(X^2 + 2)v = (T^2 + 2I)(v) = (x_{n-1} + 2x_1, x_n + 2x_2, x_1 + 2x_3, \ldots, x_{n-2} + 2x_n).$$
◀

   (c) Compute $(X^{n-1} + X^{n-2} + \cdots + X + 1)v$.

   ▶ **Solution.** $(X^{n-1}+X^{n-2}+\cdots+X+1)v = (y, \ldots, y)$ where $y = x_1+\cdots+x_n$. ◀

   (d) If $e_1 = (1, 0, \ldots, 0)$ and $f(X) = a_0 + a_1X + \cdots + a_{n-1}X^{n-1}$ is an arbitrary polynomial of degree $\leq n-1$, compute $f(X)e_1$.

   ▶ **Solution.** $f(X)e_1 = (a_0, a_1, \ldots, a_{n-1})$. ◀

   (e) Verify that $(X^n - 1)v = 0$ for all $v \in V$.

   ▶ **Solution.** Since $T^n v = v$ for all $v \in V$, it follows that $(X^n - 1)v = v$ for all $v \in V$. ◀

   (f) Show that no nonzero polynomial $f(X)$ of degree $< n$ has the property $f(X)v = 0$ for all $v \in V$.

   ▶ **Solution.** By part (d), it follows that $f(X)e_1 \neq 0$ for any nonzero polynomial of degree $\leq n-1$. ◀

2. Let $\mathbb{F} = \mathbb{R}$ and let $V = \mathbb{R}^2$.

   (a) If $T_1 : V \to V$ is counterclockwise rotation about the origin by $\pi/2$ radians, show that V and $\{0\}$ are the only $\mathbb{R}[X]$ submodules of V_{T_1}.

▶ **Solution.** The $\mathbb{R}[x]$-submodules of V_{T_1} are the T_1-invariant subspaces of V. But the subspaces of V (other than V and $\{0\}$) are the lines through the origin, and rotation of a line through the origin by $\pi/2$ radians produces a line perpendicular to the starting line. Hence, no such line is invariant under T_1. ◀

(b) If $T_2 : V \to V$ is counterclockwise rotation about the origin by π radians, show that *every* subspace of V is an $\mathbb{R}[X]$ submodule of V_{T_2}.

▶ **Solution.** Since rotation of a line through the origin by π radians produces the same line (with individual vectors v switched to $-v$), it follows that all subspaces of V are T_2-invariant, and hence every subspace of V is an $\mathbb{R}[X]$ submodule of V_{T_2}. ◀