Do the following exercises from the text:

Pages	**Exercises**
179–181	50, 52, 58, 59
337–340	3, 13 (a), (b), (c); 22, 30, 31

These exercises are primarily related to the computation and use of the Smith normal form for matrices over a Euclidean domain, a topic that we covered in class. The material covered in class is the algorithm described in Remark 3.4, Page 309. Useful calculations, similar to examples done in class, are illustrated in Example 3.5 (Page 309) and Examples 4.5 – 4.7 (Pages 325–327).

50. Find a basis and the invariant factors for the submodule of $\mathbb{Z}^3$ generated by $x_1 = (1,\,0,\,-1)$, $x_2 = (4,\,3,\,-1)$, $x_3 = (0, 9, 3)$, and $x_4 = (3,\,12,\,3)$.

► **Solution.** Starting with the relation matrix A defined by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 4 & 3 & -1 \\ 0 & 9 & 3 \\ 3 & 12 & 3 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix},$$

where $\{e_1,\, e_2,\, e_3\}$ is the standard basis of $\mathbb{Z}^3$, A is reduced to Smith normal form by performing row and column operations:

$$QAP^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ -12 & 3 & -1 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 4 & 3 & -1 \\ 0 & 9 & 3 \\ 3 & 12 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then

$$P = (P^{-1})^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

so

$$QA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus a basis of the requested submodule is $\{v_1,\, v_2,\, v_3\}$ where

$$\begin{aligned} v_1 &= (1,\,0,\,-1) = x_1 \\ v_2 &= (0,\,3,\,3) = 3(0,\,1,\,1) = -4x_1 + x_2 \\ v_3 &= (0,\,0,\,6) = 6(0,\,0,\,1) = -12x_1 + 3x_2 - x_3. \end{aligned}$$

Thus the invariant factors are the multipliers 1, 3, and 6. ◄

52. Determine the structure of $\mathbb{Z}^3/K$ where K is generated by $x_1 = (2, 1, -3)$ and $x_2 = (1, -1, 2)$.

► **Solution.** The Smith normal form of the matrix $\begin{bmatrix} 2 & 1 & -3 \\ 1 & -1 & 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Thus there is a basis e_1', e_2', e_3' of $\mathbb{Z}^3$ such that the first two vectors are a basis of K. Thus $\mathbb{Z}^3/K \cong \mathbb{Z}$. ◄

58, 59. Use elementary divisors (invariant factors) to describe all abelian groups of order 144 and 168.

► **Solution.** All abelian groups of a given order $m = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ can be described uniquely in elementary divisor form by listing the distinct ways of writing each r_i as a sum of nondecreasing natural numbers ≥ 1, and then the invariant factor decomposition can be computed from the elementary divisor form. (See Example 7.23, Page 169.) We describe this process for $m = 144$ and $m = 168$.

(a) $144 = 2^4 \cdot 3^2$ so $p_1 = 2$, $r_1 = 4$, $p_2 = 3$, $r_2 = 2$.

Abelian Groups of order 144

Exponent Partition		Group	
$r_1 = 4$	$r_2 = 2$	Elementary Divisor Form	Invariant Factor Form
4	2	$\mathbb{Z}_{2^4} \times \mathbb{Z}_{3^2}$	$\mathbb{Z}_{144}$
$3+1$	2	$\mathbb{Z}_{2^3} \times \mathbb{Z}_2 \times \mathbb{Z}_{3^2}$	$\mathbb{Z}_2 \times \mathbb{Z}_{72}$
$2+2$	2	$\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2} \times \mathbb{Z}_{3^2}$	$\mathbb{Z}_4 \times \mathbb{Z}_{36}$
$2+1+1$	2	$\mathbb{Z}_{2^2} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{3^2}$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{36}$
$1+1+1+1$	2	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{3^2}$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{18}$
4	$1+1$	$\mathbb{Z}_{2^4} \times \mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_3 \times \mathbb{Z}_{48}$
$3+1$	$1+1$	$\mathbb{Z}_{2^3} \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_6 \times \mathbb{Z}_{24}$
$2+2$	$1+1$	$\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2} \times \mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_{12} \times \mathbb{Z}_{12}$
$2+1+1$	$1+1$	$\mathbb{Z}_{2^2} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_{12}$
$1+1+1+1$	$1+1$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_6$

(b) $168 = 2^3 \cdot 3 \cdot 7$ so $p_1 = 2$, $r_1 = 3$, $p_2 = 3$, $r_2 = 1$, and $p_3 = 3$, $r_3 = 1$.

Abelian Groups of order 168

Exponent Partition		Group	
$r_1 = 3$	$r_2 = r_3 = 1$	Elementary Divisor Form	Invariant Factor Form
3	1	$\mathbb{Z}_{2^3} \times \mathbb{Z}_3 \times \mathbb{Z}_7$	$\mathbb{Z}_{168}$
$2+1$	1	$\mathbb{Z}_{2^2} \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_7$	$\mathbb{Z}_2 \times \mathbb{Z}_{84}$
$1+1+1$	1	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times Z_7$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{42}$

◄

3. Let M be an abelian group with three generators v_1, v_2, and v_3, subject to the relations

$$\begin{aligned} 2v_1 - \ \ 4v_2 - 2v_3 &= 0 \\ 10v_1 - \ \ 6v_2 + 4v_3 &= 0 \\ 6v_1 - 12v_2 - 6v_3 &= 0. \end{aligned}$$

Assuming the matrix identity

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 10 & 6 \\ -4 & -6 & -12 \\ -2 & 4 & -6 \end{bmatrix} \begin{bmatrix} 1 & -5 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

show that $M \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{14} \oplus \mathbb{Z}$, and find new generators w_1, w_2, and w_3 such that $2w_1 = 0$, $14w_2 = 0$, and w_3 has infinite order.

▶ **Solution.** The description of M means that M is the image of the $\mathbb{Z}$-module homomorphism $\eta : \mathbb{Z}^3 \to M$ determined by $\eta(e_i) = v_i$ with $K = \operatorname{Ker}(\eta)$ generated by the three vectors $f_1 = 2e_1 - 4e_2 - 2e_3$, $f_2 = 10e_1 - 6e_2 + 4e_3$, and $f_3 = 6e_1 - 12e_2 - 6e_3$. The relation matrix A relating the generators f_1, f_2, f_3 to the basis e_1, e_2, e_3 of $\mathbb{Z}^3$ is

$$A = \begin{bmatrix} 2 & -4 & -2 \\ 10 & -6 & 4 \\ 6 & -12 & -6 \end{bmatrix}.$$

Notice that this matrix is the transpose of the central matrix in the stated matrix product. Thus, taking the transpose of the given matrix product, and recalling that the transpose reverses the order of products, i.e., $(BC)^T = C^T B^T$, we get a matrix identity

$$QAP = \operatorname{diag}(2,\, 14,\, 0),$$

where

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Define a new basis e_1', e_2', e_3' of $\mathbb{Z}^3$ by the equation

$$\begin{bmatrix} e_1' \\ e_2' \\ e_3' \end{bmatrix} = P^{-1} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} e_1 - 2e_2 - e_3 \\ e_2 + e_3 \\ e_3 \end{bmatrix},$$

and new generators f_1', f_2', f_3' on K by the equation

$$\begin{bmatrix} f_1' \\ f_2' \\ f_3' \end{bmatrix} = Q \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ -5f_1 + f_2 \\ -3f_1 + f_3 \end{bmatrix}.$$

Then

$$\begin{bmatrix} f_1' \\ f_2' \\ f_3' \end{bmatrix} = Q \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = QA \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = QAPP^{-1} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 0 \end{bmatrix} P^{-1} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 2e_1' \\ 14e_2' \\ 0 \end{bmatrix}.$$

Hence $K = \mathbb{Z}(2e_1') \oplus \mathbb{Z}(14e_2')$ so that

$$\mathbb{Z}^3/K \cong (\mathbb{Z}e_1' \oplus \mathbb{Z}e_2' \oplus \mathbb{Z}e_3')/(\mathbb{Z}(2e_1') \oplus \mathbb{Z}(14e_2')) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{14} \oplus \mathbb{Z},$$

with corresponding generators $w_1 = \eta(e_1') = v_1 - 2v_2 - v_3$, $w_2 = \eta(e_2') = v_2 + v_3$, and $w_3 = \eta(e_3') = v_3$. ◀

13. Find the Smith normal form for each of the following matrices:

(a) $\begin{bmatrix} -2 & 0 & 10 \\ 0 & -3 & -4 \\ 1 & 2 & -1 \end{bmatrix} \in M_3(\mathbb{Z})$. **Answer:** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{bmatrix}$.

(b) $\begin{bmatrix} 2 & 6 & -8 \\ 12 & 14 & 6 \\ 4 & -4 & 8 \end{bmatrix} \in M_3(\mathbb{Z})$.

▶ **Solution.** The elementary row and column operations needed will not be listed explicitly. They should be deducible from the results.

$$\begin{bmatrix} 2 & 6 & -8 \\ 12 & 14 & 6 \\ 4 & -4 & 8 \end{bmatrix} \mapsto \begin{bmatrix} 2 & 6 & -8 \\ 0 & -22 & 54 \\ 0 & -16 & 24 \end{bmatrix} \mapsto \begin{bmatrix} 2 & 0 & 0 \\ 0 & -22 & 54 \\ 0 & -16 & 24 \end{bmatrix}$$

$$\mapsto \begin{bmatrix} 2 & 0 & 0 \\ 0 & 16 & -24 \\ 0 & -22 & 54 \end{bmatrix} \mapsto \begin{bmatrix} 2 & 0 & 0 \\ 0 & 16 & -24 \\ 0 & 10 & 6 \end{bmatrix} \mapsto \begin{bmatrix} 2 & 0 & 0 \\ 0 & 10 & 6 \\ 0 & 16 & -24 \end{bmatrix}$$

$$\mapsto \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 10 \\ 0 & -24 & 16 \end{bmatrix} \mapsto \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 10 \\ 0 & 0 & 56 \end{bmatrix} \mapsto \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 4 \\ 0 & 0 & 56 \end{bmatrix}$$

$$\mapsto \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 6 \\ 0 & 56 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 56 & -56 \end{bmatrix} \mapsto \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & -56 & 56 \end{bmatrix}$$

$$\mapsto \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 168 \end{bmatrix}$$

The last matrix is in Smith normal form. ◀

(c) $\begin{bmatrix} X(X-1)^3 & 0 & 0 \\ 0 & (X-1) & 0 \\ 0 & 0 & X \end{bmatrix} \in M_3(\mathbb{Q}[X])$.

▶ **Solution.** Begin by exchanging the first and third rows, followed by exchanging the first and third columns to put the X term in the upper left hand corner.

Then proceed with row and column operations as follows:

$$\begin{bmatrix} X & 0 & 0 \\ 0 & (X-1) & 0 \\ 0 & 0 & X(X-1)^3 \end{bmatrix} \mapsto \begin{bmatrix} X & X-1 & 0 \\ 0 & X-1 & 0 \\ 0 & 0 & X(X-1)^3 \end{bmatrix}$$

$$\mapsto \begin{bmatrix} X & -1 & 0 \\ 0 & X-1 & 0 \\ 0 & 0 & X(X-1)^3 \end{bmatrix} \mapsto \begin{bmatrix} -1 & X & 0 \\ X-1 & 0 & 0 \\ 0 & 0 & X(X-1)^3 \end{bmatrix}$$

$$\mapsto \begin{bmatrix} -1 & 0 & 0 \\ X-1 & X(X-1) & 0 \\ 0 & 0 & X(X-1)^3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & X(X-1) & 0 \\ 0 & 0 & X(X-1)^3 \end{bmatrix}$$

The last matrix is in Smith normal form. ◀

22. Find all integral solutions to the following systems $AX = B$ of equations:

(a) $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$

▶ **Solution.** The strategy is explained in Example 4.6, Page 326. Compute the Smith normal form of A:

$$QAP^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & -2 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then the equation $AX = B$ becomes $A(P^{-1})PX = B$ or $APY = B$ where $Y = PX$ so that

$$AP^{-1}Y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

Thus,

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ t \end{bmatrix},$$

where $t \in \mathbb{Z}$ is arbitrary. Therefore,

$$X = P^{-1}Y = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ t \end{bmatrix} = \begin{bmatrix} 5-2t \\ 1-t \\ t \end{bmatrix},$$

where $t \in \mathbb{Z}$ is arbitrary. ◀

(b) $A = \begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 5 \\ 1 \\ 7 \end{bmatrix}$. **Answer:** $X = \begin{bmatrix} 1+t \\ t \\ -5+2t \end{bmatrix}$, $t \in Z$ arbitrary.

(c) $A = \begin{bmatrix} 8 & 19 & 30 \\ 6 & 14 & 22 \end{bmatrix}, \quad B = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$. **Answer:** No solutions in integers.

30. Suppose that an abelian group G has generators x_1, x_2, and x_3 subject to the relations $x_1 - 3x_3 = 0$ and $x_1 + 2x_2 + 5x_3 = 0$. Determine the invariant factors of G and $|G|$ is G is finite.

► **Solution.** The relations matrix for G is $A = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}$ which has Smith normal form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$. This implies that $G \cong \mathbb{Z}_2 \times \mathbb{Z}$ so G is infinite and has one invariant factor, namely 2. ◄

31. Suppose that an abelian group G has generators x_1, x_2, and x_3 subject to the relations $2x_1 - x_2 = 0$, $x_1 - 3x_2$, and $x_1 + x_2 + x_3 = 0$. Determine the invariant factors of G and $|G|$ is G is finite.

► **Solution.** The relations matrix for G is $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & -3 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ which has Smith normal form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. This implies that $G \cong \mathbb{Z}_5$ so G is cyclic of order 5 and hence has one invariant factor, namely 5. ◄