The following exercises are intended solely to be representative of the type of questions that might be expected on the second exam, which covers the ring theory from Chapter 2 of the text, plus the language of module theory as presented in sections 3.1 to 3.3.

1. Let $R = \mathbb{Z}[\sqrt{2}]$ and let $I = \langle 2 \rangle$ be the principal ideal generated by $\sqrt{2}$.
   (a) Verify that $I = \{a + b\sqrt{2} \in \mathbb{Z}[\sqrt{2}] : \ a \text{ is even}\}$.
   (b) Define $\varphi : \mathbb{Z}[\sqrt{2}] \to \mathbb{Z}_2$ by $\varphi(a + b\sqrt{2}) = a \pmod 2$. Show that φ is a ring homomorphism such that $\mathrm{Ker}(\varphi) = I$.
   (c) Verify that I is a maximal ideal of R.
2. (a) Show that the ring $\mathbb{Z}[\sqrt{-1}]$ is a Euclidean ring using the value function
   $$v(m + n\sqrt{-1}) = m^2 + n^2.$$
   (b) If $a = 4 + 3\sqrt{-1}$ and $b = 3 - \sqrt{-1}$, write $a = qb + r$ where $r = 0$ or $v(r) < v(b)$.
3. Let S be a subring of a commutative ring R with identity. We assume that the identity of R is also the identity of S. Let $I \subseteq R$ be an ideal and let $T = R/I$ with $\pi : R \to T$ the natural projection map, i.e., $\pi(a) = a + I$. Answer the following questions concerning the relationships between these rings. If the answer is yes, prove it. Otherwise, give a counterexample. All answers are intended to be very brief.
   (a) If $a \in S$ and a is a unit of R, is a a unit of S?
   (b) If $b \in R$ is a unit, is $\pi(b)$ a unit of T?
   (c) If R is an integral domain, is T an integral domain?
   (d) If R is a PID (principal ideal domain), is every ideal of T principal?
4. (a) Let R be a UFD (Unique Factorization Domain) and let $I \neq 0$ be a non-zero *prime* ideal. Show that there is a principal prime ideal $\langle a \rangle$ of R such that $\langle 0 \rangle \neq \langle a \rangle \subseteq I$. (*Hint:* Factor a non-zero element of I.)
   (b) Suppose that J is an ideal of $\mathbb{Z}$ with $J \neq \mathbb{Z}$, and suppose that J is not prime. Show that J does not contain a non-zero principal prime ideal.
5. Let $R = \mathbb{Z}[\sqrt{-3}] = \{a + b\sqrt{-3} : a, b \in \mathbb{Z}\}$.
   (a) Why is R an integral domain?
   (b) What are the units in R?
   (c) Is the element 2 irreducible in R?
   (d) If x, $y \in R$, and 2 divides xy, does it follow that 2 divides either x or y? Justify your answer.
6. (a) For which fields $\mathbb{Z}_p$ is $X^2 + 1$ a factor of $f(X) = X^3 + X^2 + 106X + 71$ in $\mathbb{Z}_p[X]$?
   (b) For these fields $\mathbb{Z}_p$, what is the prime factorization of $f(X)$ in $\mathbb{Z}_p[X]$?

7. Let $R = \mathbb{Z}[X]$. Answer the following questions about the ring R. You may quote an appropriate theorem, provide a counterexample, or give a short proof to justify your answer.

   (a) Is R a unique factorization domain?

   (b) Is R a principal ideal domain?

   (c) Find the group of units of R.

   (d) Find a prime ideal of R which is not maximal.

   (e) Find a maximal ideal of R.

8. An element a in a ring R is *nilpotent* if $a^n = 0$ for some natural number n.

   (a) If R is a commutative ring with identity, show that the set of nilpotent elements forms an ideal.

   (b) Describe all of the nilpotent elements in the ring $\mathbb{C}[X]/\langle f(X)\rangle$, where

   $$f(X) = (X-1)(X^2-1)(X^3-1).$$

   (c) Show that part (a) need not be true if R is not commutative. (*Hint:* Try a matrix ring.)

9. Suppose that R is an integral domain and X is an indeterminate.

   (a) Prove that if R is a field, then the polynomial ring $R[X]$ is a PID (principal ideal domain).

   (b) Show, conversely, that if $R[X]$ is a PID, then R is a field.

10. Let $\mathbb{Z}\left[\frac{1}{2}\right]$ denote the subring of $\mathbb{Q}$ generated by $\mathbb{Z}$ and $\frac{1}{2}$. Is $\mathbb{Z}\left[\frac{1}{2}\right]$ finitely generated as a $\mathbb{Z}$-module? Justify your answer.

11. Let N be a submodule of an R-module M. Show that if N and M/N are finitely generated, then M is finitely generated.