The following exercises are intended solely to be representative of what might be expected on the first exam, which covers the group theory from Chapter 1 of the text.

1. Prove that a group G is abelian if and only if the inversion map $\varphi : G \to G$ defined by $\varphi(a) = a^{-1}$ is a group homomorphism.

2. (a) Compute $\alpha\beta\alpha^{-1}$ where $\alpha = (1\ 3\ 5)(1\ 2)$ and $\beta = (1\ 5\ 7\ 9)$.
   (b) Given $\gamma = (1\ 2)(3\ 4)$ and $\delta = (5\ 6)(1\ 3)$ find an element $\sigma \in S_6$ with $\sigma\gamma\sigma^{-1} = \delta$.
   (c) Find the number of conjugates of $\gamma = (1\ 2)(3\ 4)$ in S_6.
   (d) Find the number of conjugates of $\gamma = (1\ 2)(3\ 4)$ in A_6.

3. Let G be a group of order 28.
   (a) Prove that G has a normal subgroup of order 7.
   (b) Prove that if G has a normal subgroup of order 4, then G is abelian.
   (c) Give an example of a nonabelian group of order 28.

4. Let $G = \langle a \rangle$ be a cyclic group of order $2n$.
   (a) What is the order of a^n? Prove that your answer is correct.
   (b) What is the order of a^2? Prove that your answer is correct.
   (c) For which n is it true that G is isomorphic to the direct product of $H = \langle a^2 \rangle$ and $K = \langle a^n \rangle$? Justify your answer.

5. Let N be a normal subgroup of order p of a finite p-group G. Prove that N is contained in the center of G. (*Hint:* The class equation may be of use.)

6. In a non-abelian group of order 55, find the number of elements of order n for $n = 1$, 5, 11, and 55.

7. If H is a normal subgroup of G such that H and G/H are both abelian, is it true that G must also be abelian? Either prove that this is true or provide a counterexample.

8. Suppose that G is a group with a normal subgroup N such that $G/N \cong \mathbb{Z}_{36}$. How many subgroups H of G are there with $N \subsetneqq H \subsetneqq G$?