

**Theorem 1.** Let  $g$  be an element of a group  $G$  and write

$$\langle g \rangle = \{g^k : k \in \mathbb{Z}\}.$$

Then  $\langle g \rangle$  is a subgroup of  $G$ .

*Proof.* Since  $1 = g^0$ ,  $1 \in \langle g \rangle$ . Suppose  $a, b \in \langle g \rangle$ . Then  $a = g^k$ ,  $b = g^m$  and  $ab = g^k g^m = g^{k+m}$ . Hence  $ab \in \langle g \rangle$  (note that  $k + m \in \mathbb{Z}$ ). Moreover,  $a^{-1} = (g^k)^{-1} = g^{-k}$  and  $-k \in \mathbb{Z}$ , so that  $a^{-1} \in \langle g \rangle$ . Thus, we have checked the three conditions necessary for  $\langle g \rangle$  to be a subgroup of  $G$ .  $\square$

**Definition 2.** If  $g \in G$ , then the subgroup  $\langle g \rangle = \{g^k : k \in \mathbb{Z}\}$  is called the **cyclic subgroup of  $G$  generated by  $g$** . If  $G = \langle g \rangle$ , then we say that  $G$  is a **cyclic group** and that  $g$  is a **generator** of  $G$ .

**Examples 3.** 1. If  $G$  is any group then  $\{1\} = \langle 1 \rangle$  is a cyclic subgroup of  $G$ .

2. The group  $G = \{1, -1, i, -i\} \subseteq \mathbb{C}^*$  (the group operation is multiplication of complex numbers) is cyclic with generator  $i$ . In fact  $\langle i \rangle = \{i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i\} = G$ . Note that  $-i$  is also a generator for  $G$  since  $\langle -i \rangle = \{(-i)^0 = 1, (-i)^1 = -i, (-i)^2 = -1, (-i)^3 = i\} = G$ . Thus a cyclic group may have more than one generator. However, not all elements of  $G$  need be generators. For example  $\langle -1 \rangle = \{1, -1\} \neq G$  so  $-1$  is not a generator of  $G$ .

3. The group  $G = \mathbb{Z}_7^*$  is the group of units of the ring  $\mathbb{Z}_7$  is a cyclic group with generator 3. Indeed,

$$\langle 3 \rangle = \{1 = 3^0, 3 = 3^1, 2 = 3^2, 6 = 3^3, 4 = 3^4, 5 = 3^5\} = G.$$

Note that 5 is also a generator of  $G$ , but that  $\langle 2 \rangle = \{1, 2, 4\} \neq G$  so that 2 is not a generator of  $G$ .

4.  $G = \langle \pi \rangle = \{\pi^k : k \in \mathbb{Z}\}$  is a cyclic subgroup of  $\mathbb{R}^*$ .

5. The group  $G = \mathbb{Z}_8^*$  is not cyclic. Indeed, since  $\mathbb{Z}_8^* = \{1, 3, 5, 7\}$  and  $\langle 1 \rangle = \{1\}$ ,  $\langle 3 \rangle = \{1, 3\}$ ,  $\langle 5 \rangle = \{1, 5\}$ ,  $\langle 7 \rangle = \{1, 7\}$ , it follows that  $\mathbb{Z}_8^* \neq \langle a \rangle$  for any  $a \in \mathbb{Z}_8^*$ .

If a group  $G$  is written additively, then the identity element is denoted 0, the inverse of  $a \in G$  is denoted  $-a$ , and the powers of  $a$  become  $na$  in additive notation. Thus, with this notation, the cyclic subgroup of  $G$  generated by  $a$  is  $\langle a \rangle = \{na : n \in \mathbb{Z}\}$ , consisting of all the multiples of  $a$ . Among groups that are normally written additively, the following are two examples of cyclic groups.

6. The integers  $\mathbb{Z}$  are a cyclic group. Indeed,  $\mathbb{Z} = \langle 1 \rangle$  since each integer  $k = k \cdot 1$  is a multiple of 1, so  $k \in \langle 1 \rangle$  and  $\langle 1 \rangle = \mathbb{Z}$ . Also,  $\mathbb{Z} = \langle -1 \rangle$  because  $k = (-k) \cdot (-1)$  for each  $k \in \mathbb{Z}$ .

7.  $\mathbb{Z}_n$  is a cyclic group under addition with generator 1.

**Theorem 4.** Let  $g$  be an element of a group  $G$ . Then there are two possibilities for the cyclic subgroup  $\langle g \rangle$ .

**Case 1:** The cyclic subgroup  $\langle g \rangle$  is finite. In this case, there exists a smallest positive integer  $n$  such that  $g^n = 1$  and we have

(a)  $g^k = 1$  if and only if  $n|k$ .

(b)  $g^k = g^m$  if and only if  $k \equiv m \pmod{n}$ .

(c)  $\langle g \rangle = \{1, g, g^2, \dots, g^{n-1}\}$  and the elements  $1, g, g^2, \dots, g^{n-1}$  are distinct.

**Case 2:** The cyclic subgroup  $\langle g \rangle$  is infinite. Then

(d)  $g^k = 1$  if and only if  $k = 0$ .

(e)  $g^k = g^m$  if and only if  $k = m$ .

(f)  $\langle g \rangle = \{\dots, g^{-3}, g^{-2}, g^{-1}, 1, g, g^2, g^3, \dots\}$  and all of these powers of  $g$  are distinct.

*Proof.* **Case 1.** Since  $\langle g \rangle$  is finite, the powers  $g, g^2, g^3, \dots$  are not all distinct, so let  $g^k = g^m$  with  $k < m$ . Then  $g^{m-k} = 1$  where  $m - k > 0$ . Hence there is a positive integer  $l$  with  $g^l = 1$ . Hence there is a smallest such positive integer. We let  $n$  be this smallest positive integer, i.e.,  $n$  is the smallest positive integer such that  $g^n = 1$ .

(a) If  $n|k$  then  $k = qn$  for some  $q \in \mathbb{Z}$ . Then  $g^k = g^{qn} = (g^n)^q = 1^q = 1$ . Conversely, if  $g^k = 1$ , use the division algorithm to write  $k = qn + r$  with  $0 \leq r < n$ . Then  $g^r = g^k(g^n)^{-q} = 1(1)^{-q} = 1$ . Since  $r < n$ , this contradicts the minimality of  $n$  unless  $r = 0$ . Hence  $r = 0$  and  $k = qn$  so that  $n|k$ .

(b)  $g^k = g^m$  if and only if  $g^{k-m} = 1$ . Now apply Part (a).

(c) Clearly,  $\{1, g, g^2, \dots, g^{n-1}\} \subseteq \langle g \rangle$ . To prove the other inclusion, let  $a \in \langle g \rangle$ . Then  $a = g^k$  for some  $k \in \mathbb{Z}$ . As in Part (a), use the division algorithm to write  $k = qn + r$ , where  $0 \leq r < n$ . Then

$$a = g^k = g^{qn+r} = (g^n)^q g^r = 1^q g^r = g^r \in \{1, g, g^2, \dots, g^{n-1}\}$$

which shows that  $\langle g \rangle \subseteq \{1, g, g^2, \dots, g^{n-1}\}$ , and hence that

$$\langle g \rangle = \{1, g, g^2, \dots, g^{n-1}\}.$$

Finally, suppose that  $g^k = g^m$  where  $0 \leq k < m \leq n-1$ . Then  $g^{m-k} = 1$  and  $0 < m-k < n$ . This implies that  $m-k = 0$  because  $n$  is the smallest positive power of  $g$  which equals 1. Hence all of the elements  $1, g, g^2, \dots, g^{n-1}$  are distinct.

**Case 2.** (d) Certainly,  $g^k = 1$  if  $k = 0$ . If  $g^k = 1$ ,  $k \neq 0$ , then  $g^{-k} = (g^k)^{-1} = 1^{-1} = 1$ , also. Hence  $g^n = 1$  for some  $n > 0$ , which implies that  $\langle g \rangle$  is finite by the proof of Part (c), contrary to our hypothesis in Case 2. Thus  $g^k = 1$  implies that  $k = 0$ .

(e)  $g^k = g^m$  if and only if  $g^{k-m} = 1$ . Now apply Part (d).

(f)  $\langle g \rangle = \{g^k : k \in \mathbb{Z}\}$  by definition of  $\langle g \rangle$ , so all that remains is to check that these powers are distinct. But this is the content of Part (e).  $\square$

Recall that if  $g$  is an element of a group  $G$ , then the **order** of  $g$  is the smallest positive integer  $n$  such that  $g^n = 1$ , and it is denoted  $o(g) = n$ . If there is no such positive integer, then we say that  $g$  has **infinite order**, denoted  $o(g) = \infty$ . By Theorem 4, the concept of order of an element  $g$  and order of the cyclic subgroup generated by  $g$  are the same.

**Corollary 5.** If  $g$  is an element of a group  $G$ , then  $o(g) = |\langle g \rangle|$ .

*Proof.* This is immediate from Theorem 4, Part (c).  $\square$

If  $G$  is a cyclic group of order  $n$ , then it is easy to compute the order of all elements of  $G$ . This is the content of the following result.

**Theorem 6.** Let  $G = \langle g \rangle$  be a cyclic group of order  $n$ , and let  $0 \leq k < n$ . If  $m = \gcd(k, n)$ , then  $o(g^k) = \frac{n}{m}$ .

*Proof.* Let  $k = ms$  and  $n = mt$ . Then  $(g^k)^{n/m} = g^{kn/m} = g^{msn/m} = (g^n)^s = 1^s = 1$ . Hence  $n/m$  divides  $o(g^k)$  by Theorem 4 Part (a). Now suppose that  $(g^k)^r = 1$ . Then  $g^{kr} = 1$ , so by Theorem 3 Part (a),  $n \mid kr$ . Hence

$$\frac{n}{m} \mid \frac{k}{m}r$$

and since  $n/m$  and  $k/m$  are relatively prime, it follows that  $n/m$  divides  $r$ . Hence  $n/m$  is the smallest power of  $g^k$  which equals 1, so  $o(g^k) = n/m$ .  $\square$

**Theorem 7.** *Let  $G = \langle g \rangle$  be a cyclic group where  $o(g) = n$ . Then  $G = \langle g^k \rangle$  if and only if  $\gcd(k, n) = 1$ .*

*Proof.* By Theorem 6, if  $m = \gcd(k, n)$ , then  $o(g^k) = n/m$ . But  $G = \langle g^k \rangle$  if and only if  $o(g^k) = |G| = n$  and this happens if and only if  $m = 1$ , i.e., if and only if  $\gcd(k, n) = 1$ .  $\square$

**Example 8.** If  $G = \langle g \rangle$  is a cyclic group of order 12, then the generators of  $G$  are the powers  $g^k$  where  $\gcd(k, 12) = 1$ , that is  $g, g^5, g^7$ , and  $g^{11}$ . In the particular case of the additive cyclic group  $\mathbb{Z}_{12}$ , the generators are the integers 1, 5, 7, 11 (mod 12).

Now we ask what the subgroups of a cyclic group look like. The question is completely answered by Theorem 10. Theorem 9 is a preliminary, but important, result.

**Theorem 9.** *Every subgroup of a cyclic group is cyclic.*

*Proof.* Suppose that  $G = \langle g \rangle = \{g^k : k \in \mathbb{Z}\}$  is a cyclic group and let  $H$  be a subgroup of  $G$ . If  $H = \{1\}$ , then  $H$  is cyclic, so we assume that  $H \neq \{1\}$ , and let  $g^k \in H$  with  $g^k \neq 1$ . Then, since  $H$  is a subgroup,  $g^{-k} = (g^k)^{-1} \in H$ . Therefore, since  $k$  or  $-k$  is positive,  $H$  contains a positive power of  $g$ , not equal to 1. So let  $m$  be the smallest positive integer such that  $g^m \in H$ . Then, certainly all powers of  $g^m$  are also in  $H$ , so we have  $\langle g^m \rangle \subseteq H$ . We claim that this inclusion is an equality. To see this, let  $g^k$  be any element of  $H$  (recall that all elements of  $G$ , and hence  $H$ , are powers of  $g$  since  $G$  is cyclic). By the division algorithm, we may write  $k = qm + r$  where  $0 \leq r < m$ . But  $g^k = g^{qm+r} = g^{qm}g^r = (g^m)^qg^r$  so that

$$g^r = (g^m)^{-q}g^k \in H.$$

Since  $m$  is the smallest positive integer with  $g^m \in H$  and  $0 \leq r < m$ , it follows that we must have  $r = 0$ . Then  $g^k = (g^m)^q \in \langle g^m \rangle$ . Hence we have shown that  $H \subseteq \langle g^m \rangle$  and hence  $H = \langle g^m \rangle$ . That is  $H$  is cyclic with generator  $g^m$  where  $m$  is the smallest positive integer for which  $g^m \in H$ .  $\square$

**Theorem 10 (Fundamental Theorem of Finite Cyclic Groups).** *Let  $G = \langle g \rangle$  be a cyclic group of order  $n$ .*

1. *If  $H$  is any subgroup of  $G$ , then  $H = \langle g^d \rangle$  for some  $d \mid n$ .*
2. *If  $H$  is any subgroup of  $G$  with  $|H| = k$ , then  $k \mid n$ .*
3. *If  $k \mid n$ , then  $\langle g^{n/k} \rangle$  is the unique subgroup of  $G$  of order  $k$ .*

*Proof.* 1. By Theorem 9,  $H$  is a cyclic group and since  $|G| = n < \infty$ , it follows that  $H = \langle g^m \rangle$  where  $m > 0$ . Let  $d = \gcd(m, n)$ . Since  $d \mid n$  it is sufficient to show that  $H = \langle g^d \rangle$ . But  $d \mid m$  also, so  $m = qd$ . Then  $g^m = (g^d)^q$  so  $g^m \in \langle g^d \rangle$ . Hence  $H = \langle g^m \rangle \subseteq \langle g^d \rangle$ . But  $d = rm + sn$ , where  $r, s \in \mathbb{Z}$ , so

$$g^d = g^{rm+sn} = g^{rm}g^{sn} = (g^m)^r(g^n)^s = (g^m)^r(1)^s = (g^m)^r \in \langle g^m \rangle = H.$$

This shows that  $\langle g^d \rangle \subseteq H$  and hence  $\langle g^d \rangle = H$ .

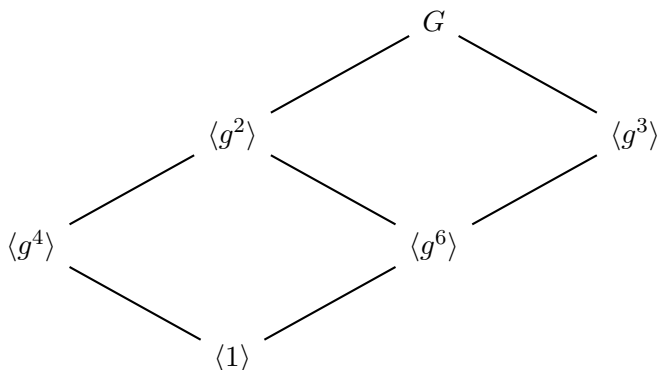
2. By Part (a),  $H = \langle g^d \rangle$  where  $d \mid n$ . Then  $k = |H| = n/d$  so  $k \mid n$ .
3. Suppose that  $K$  is any subgroup of  $G$  of order  $k$ . By Part (a), let  $K = \langle g^m \rangle$  where  $m \mid n$ . Then Theorem 6 gives  $k = |K| = |g^m| = n/m$ . Hence  $m = n/k$ , so  $K = \langle g^{n/k} \rangle$ . This proves (c). □

**Remark 11.** Part (b) of Theorem 10 is actually true for *any* finite group  $G$ , whether or not it is cyclic. This result is Lagrange's Theorem (Theorem 5.2.3, Page 219 of your text).

The subgroups of a group  $G$  can be diagrammatically illustrated by listing the subgroups, and indicating inclusion relations by means of a line directed upward from  $H$  to  $K$  if  $H$  is a subgroup of  $K$ . Such a scheme is called the **lattice diagram** for the subgroups of the group  $G$ . We will illustrate by determining the lattice diagram for all the subgroups of a cyclic group  $G = \langle g \rangle$  of order 12. Since the order of  $g$  is 12, Theorem 10 (c) shows that there is exactly one subgroup  $\langle g^d \rangle$  for each divisor  $d$  of 12. The divisors of 12 are 1, 2, 3, 4, 6, 12. Then the unique subgroup of  $G$  of each of these orders is, respectively,

$$\{1\} = \langle g^{12} \rangle, \quad \langle g^6 \rangle, \quad \langle g^4 \rangle, \quad \langle g^3 \rangle, \quad \langle g^2 \rangle, \quad \langle g \rangle = G.$$

Note that  $\langle g^m \rangle \subseteq \langle g^k \rangle$  if and only if  $k \mid m$ . Hence the lattice diagram of  $G$  is:



**Exercises**

1. Find all generators of the cyclic group  $G = \langle g \rangle$  if:  
(a)  $o(g) = 5$       (b)  $o(g) = 10$       (c)  $|G| = 16$       (d)  $|G| = 20$
2. Find all generators of:  
(a)  $\mathbb{Z}_5$       (b)  $\mathbb{Z}_{10}$       (c)  $\mathbb{Z}_{16}$       (d)  $\mathbb{Z}_{20}$
3. In each case determine whether  $G$  is cyclic.  
(a)  $G = \mathbb{Z}_7^*$       (b)  $G = \mathbb{Z}_{12}^*$       (c)  $G = \mathbb{Z}_{16}^*$       (d)  $G = \mathbb{Z}_{11}^*$
4. Let  $o(g) = 20$  in a group  $G$ . Compute:  
(a)  $o(g^2)$       (b)  $o(g^8)$       (c)  $o(g^5)$       (d)  $o(g^3)$
5. In each case find all the subgroups of  $G = \langle g \rangle$  and draw the lattice diagram.  
(a)  $o(g) = 8$       (b)  $o(g) = 10$       (c)  $o(g) = 18$   
(d)  $o(g) = p^3$ , where  $p$  is prime.  
(e)  $o(g) = pq$ , where  $p$  and  $q$  are distinct primes.  
(f)  $o(g) = p^2q$ , where  $p$  and  $q$  are distinct primes.
6. (a) If  $gh = hg$  in a group and if both  $o(g)$  and  $o(h)$  are finite, show that  $o(gh)$  is finite.  
(b) In the group  $\text{GL}(2, \mathbb{Z})$ , show that  $g = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  has order 4,  $h = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$  has order 3, but the product  $gh$  has infinite order. Does this example contradict the statement of Part (a)?
7. In each case, find the subgroup  $H = \langle x, y \rangle$  of  $G$ .  
(a)  $G = \langle a \rangle$  is cyclic,  $x = a^4$ ,  $y = a^3$ .  
(b)  $G = \langle a \rangle$  is cyclic,  $x = a^6$ ,  $y = a^8$ .  
(c)  $G = \langle a \rangle$  is cyclic,  $x = a^m$ ,  $y = a^k$ ,  $\gcd(m, k) = d$ .  
(d)  $G = S(3)$ ,  $x = (1, 2)$ ,  $y = (2, 3)$ .