Instructions. The exam is open book and open notes.

1. Let $f(X) \in \mathbb{Q}[X]$ be an irreducible polynomial of degree 3. Suppose that $f(X)$ has exactly one real root α, and let $K \subseteq \mathbb{C}$ be a splitting field of $f(X)$.
   (a) Is $\mathbb{Q}(\alpha)$ a normal extension of $\mathbb{Q}$?
   (b) Is K a normal extension of $\mathbb{Q}$?
   (c) Compute $G(\mathbb{Q}(\alpha) : \mathbb{Q})$.
   (d) Compute $G(K : \mathbb{Q})$.

2. Let $f(X) = (X^2 - 2)(X^2 + 3) \in \mathbb{Q}[X]$ and let $K \subseteq \mathbb{C}$ be a splitting field of $f(X)$.
   (a) Compute the Galois group G_f of $f(X)$.
   (b) Give the complete lattice of subfields of K, including the degrees over $\mathbb{Q}$, and explain how the Fundamental Theorem of Galois theory is used to justify that you have found all of the subfields and their various inclusion relations.

3. Let $\mathbb{F}_q$ denote the finite field with q elements. For parts (a) and (b), answers are sufficient.
   (a) For which q does a finite field $\mathbb{F}_q$ exist?
   (b) What is the condition on q and t which guarantees that a finite field $\mathbb{F}_q$ is a subfield of a finite field $\mathbb{F}_t$?
   (c) If $q \neq 2$ show that the sum of all elements of $\mathbb{F}_q$ is 0.

4. (a) Let F be a field and K a field containing F. If $f(X) \in F[X]$, show that there is an isomorphism of K-algebras:
   $$K \otimes_F (F[X]/\langle f(X)\rangle) \cong K[X]/\langle f(X)\rangle.$$
   (b) By choosing F, $f(X)$, and K appropriately, find an example of two fields K and L such that the F-algebra $K \otimes_F L$ has nilpotent elements.

5. Recall that a ring R (which we assume to have an identity) is semisimple if ${}_RR$ is semisimple, that is, R is semisimple as a left R-module. Show that a semisimple ring R without zero-divisors is a division ring. (To say that R does not have zero-divisors means that if $ab = 0$ then $a = 0$ or $b = 0$.)

   Hint: The Wedderburn-Artin theorem might be of use.

6. Let $C = \langle a : a^3 = 1\rangle$ be the cyclic group of order 3 and let
   $$\mathbb{Q}C_3 = \left\{\alpha \cdot 1 + \beta \cdot a + \gamma \cdot a^2 : \alpha,\ \beta,\ \gamma \in \mathbb{Q}\right\}$$
   be the group ring of C_3 over the field $\mathbb{Q}$ of rational numbers.
   (a) Verify that $V_1 = \{c(1 + a + a^2) : c \in \mathbb{Q}\}$ and $V_2 = \{\alpha \cdot 1 + \beta \cdot a + \gamma \cdot a^2 : \alpha + \beta + \gamma = 0\}$ are $\mathbb{Q}C_3$-submodules of $\mathbb{Q}C_3$.
   (b) Show that $\mathbb{Q}C_3$ is the direct sum of V_1 and V_2. Is this the direct sum decomposition into irreducible submodules guaranteed by Maschke's theorem?