

1. Show that $x^3 + 3x + 1$ is irreducible over \mathbb{Q} and let $\theta \in \mathbb{C}$ be a root. Compute $(1 + \theta)(1 + \theta + \theta^2)$ and $\frac{1 + \theta}{1 + \theta + \theta^2}$ in $\mathbb{Q}(\theta)$.
2. Let $w = e^{\pi i/6}$ so that $w^{12} = 1$, but $w^k \neq 1$ for $1 \leq k < 12$. Find the minimal polynomial $m_{w, \mathbb{Q}}(x)$ and compute $[\mathbb{Q}[w] : \mathbb{Q}]$.
3. Compute the minimal polynomial $m_{\alpha, F}(x)$ where $\alpha = \sqrt{2} + \sqrt{5}$ and F is each of the following fields:
(a) \mathbb{Q} , (b) $\mathbb{Q}[\sqrt{5}]$, (c) $\mathbb{Q}[\sqrt{10}]$, (d) $\mathbb{Q}[\sqrt{15}]$.
4. Compute the minimal polynomial $m_{\alpha, \mathbb{Q}}(x)$ where $\alpha = \sqrt{2} + \sqrt[3]{5}$.
5. If K is a field extension of the field F and $\alpha \in K$ has minimal polynomial $f(x) \in F[x]$ of odd degree, prove that $F(\alpha) = F(\alpha^2)$. Determine whether the condition on $f(x)$ is necessary for $F(\alpha) = F(\alpha^2)$.
6. Let K be an extension field of F that is algebraic over F . Show that any subring R of K which contains F , i.e., $F \subseteq R \subseteq K$, is a field. Hence, prove that any subring of a finite dimensional extension field K/F containing F is a subfield.
7. Suppose that $K = F(\alpha)$ is a finite simple extension of the field F . Define an F -linear transformation $T_\alpha : K \rightarrow K$ by $T_\alpha(\beta) = \alpha\beta$ for all $\beta \in K$. Show that the minimal polynomial of α over F is the characteristic polynomial of T_α , that is,

$$m_{\alpha, F}(x) = \det(xI - T_\alpha).$$