1. Show that an angle of $30°$ and an angle of $15°$ cannot be trisected.

2. Let $\xi = e^{2\pi i/6} = \cos(2\pi/6) + i\sin(2\pi/6)$ be a primitive 6^{th} root of unity over $\mathbb{Q}$. Find each of the following.

   (a) The minimal polynomial $f(x) \in \mathbb{Q}[x]$ of ξ over $\mathbb{Q}$.

   (b) The splitting field F of $f(x)$ over $\mathbb{Q}$.

   (c) $[F : \mathbb{Q}]$.

3. Find a splitting field extension $K : \mathbb{Q}$ for each of the following polynomials over $\mathbb{Q}$ and in each case determine the degree $[K : \mathbb{Q}]$.

   (a) $x^4 + 1$ (b) $x^4 + 4$ (c) $(x^4 + 1)(x^4 + 4)$ (d) $(x^4 - 1)(x^4 + 4)$

4. Let $f(x) \in \mathbb{Q}[x]$ be the minimal polynomial of $\alpha = \sqrt{2 + \sqrt{2}}$.

   (a) Show that $f(x) = x^4 - 4x^2 + 2$. Thus, $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$.

   (b) Show that $\mathbb{Q}(\alpha)$ is the splitting field of $f(x)$ over $\mathbb{Q}$.

5. Let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ be the field with p elements, where p is a prime number. Write down all monic cubic polynomials in $\mathbb{F}_2[x]$, factor them completely into irreducible factors and construct a splitting field for each of them. Which of these fields are isomorphic?

6. Let $f(x) = x^3 + 2x + 2 \in \mathbb{F}_3[x]$.

   (a) Show that $f(x)$ is irreducible in $\mathbb{F}_3[x]$.

   (b) Let α be a root of $f(x)$ in some extension field K of $\mathbb{F}_3$, so that $[\mathbb{F}_3[\alpha] : \mathbb{F}_3] = \deg f(x) = 3$. Show that $\mathbb{F}_3[\alpha]$ is a splitting field of $f(x)$ over $\mathbb{F}_3$.
   Hint: Show that the map $\phi_3 : K \to K$ defined by $\phi_3(t) = t^3$ is a field homomorphism that takes a root of $f(x)$ to another root of $f(x)$.

7. Suppose that $f(x) \in F[x]$ has degree $n > 0$, and let L be the splitting field of $f(x)$ over F.

   (a) Suppose that $[L : F] = n!$. Prove that $f(x)$ is irreducible.

   (b) Show that the converse of part (a) is false.